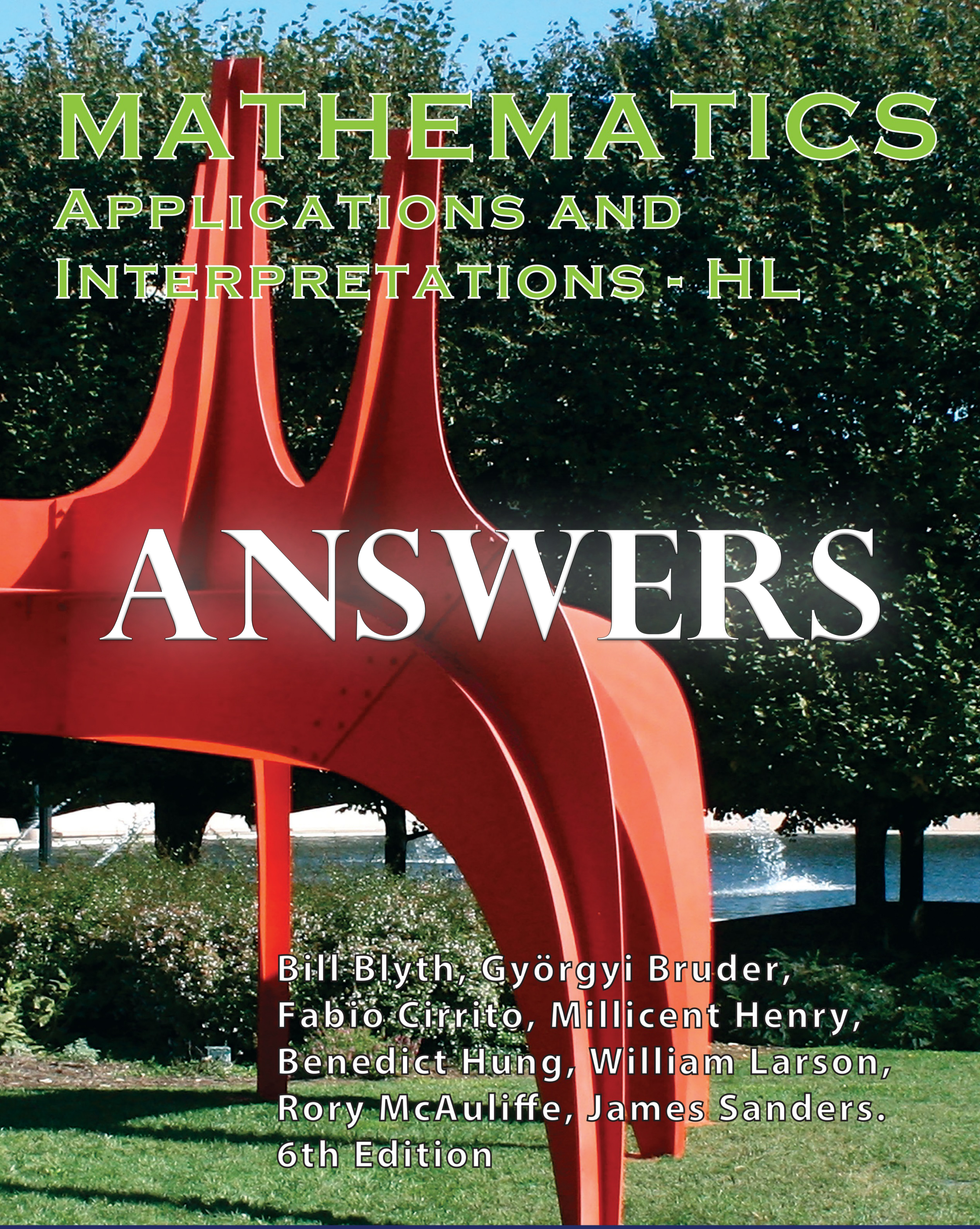

MATHEMATICS
Applications and Interpretations - HL

ANSWERS

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6th Edition

For use with the I.B. Diploma Programme

Exercise A.7.1

1	**a**	$\frac{27y^{15}}{8x^3}$	**b**	$\frac{91}{216a^6}$	**c**	2^n+2	**d**	$\frac{8x^{11}}{27y^2}$
	e	$\frac{3x^2y^2}{8}$	**f**	$3^{n+1}+3$	**g**	$4^{n+1}-4$		
	h	$2(4^{n+1}-4)$	**i**	$\frac{1-b^6}{16b^4}$				
2	**a**	64	**b**	$\left(\frac{2}{3}\right)^x$	**c**	2^{2y+1}	**d**	$\frac{1}{b^{2x}}$
	e	$\left(\frac{y}{2}\right)^6$	**f**	$\left(\frac{9}{2}\right)^{n+2}$				
3	**a**	$\frac{z^2}{xy}$	**b**	3^{7n-2}	**c**	5^{n+1}	**d**	9
	e	2^{6n+1}	**f**	2^{1-3n}	**g**	x^{2+4n-n^2}		
	h	x^{3n^2+n+1}	**i**	27				
4	$\frac{y^{2m-2}}{x^m}$							
5	**a**	-81	**b**	$-\frac{9x^8}{8y^4}$	**c**	$y-x$		
	d	$\frac{2x+1}{x+1}$	**e**	-1	**f**	$-b$		
6	**a**	$\frac{1}{x^2y^2}$	**b**	$\frac{1}{x^4}$	**c**	$-\frac{1}{x(x+h)}$		
	d	$\frac{1}{x-1}$	**e**	$\frac{1}{(x+1)(x-1)^5}$	**f**	$\frac{1}{x^2}$		
7	**a**	$118\times 5^{n-2}$	**b**	1	**c**	$\frac{b^7}{a^4}$	**d**	a^{mn}
	e	$\frac{p+q}{pq}$	**f**	$\frac{2\sqrt{a}}{a-1}$	**g**	$\frac{7}{8}$	**h**	$a^{7/8}$
8	**a**	$x^{11/12}$	**b**	$2a^{3n-2}b^{2n-2}$	**c**	2^n		
	d	$-\frac{7^{m-n}}{8}$	**e**	$\frac{6\times 5^n}{5^n+5}$				

Exercise A.7.2

1	**a**	2	**b**	-2	**c**	$\frac{2}{3}$	**d**	5	**e**	6
	f	-2.5	**g**	2	**h**	1.25	**i**	$\frac{1}{3}$		
2	**a**	-6	**b**	$-\frac{2}{3}$	**c**	-3	**d**	1.5	**e**	0.25
	f	0.25	**g**	$-\frac{1}{8}$	**h**	$-\frac{11}{4}$	**i**	-1.25		

Exercise A.7.3

1	**a**	2	**b**	2	**c**	5	**d**	3	**e**	-3
	f	-2	**g**	0	**h**	0	**i**	-1	**j**	-2
	k	0.5	**l**	-2						

2 **a** $\log_{10}10000 = 4$ **b** $\log_{10}0.001 = -3$
c $\log_{10}(x+1) = y$ **d** $\log_{10}p = 7$
e $\log_2(x-1) = y$ **f** $\log_2(y-2) = 4x$

3 **a** $2^9 = x$ **b** $b^x = y$ **c** $b^{ax} = t$
d $10^{x^2} = z$ **e** $10^{1-x} = y$ **f** $2^y = ax-b$

4 **a** 16 **b** 2 **c** 2 **d** 9 **e** $\sqrt[4]{2}$
f 125 **g** 4 **h** 9 **i** $\sqrt[3]{\frac{1}{3}}$ **j** 21 **k** 3
l 13

5 **a** 54.5982 **b** 1.3863 **c** 1.6487
d 7.3891 **e** 1.6487 **f** 0.3679
g 52.5982 **h** 4.7183 **i** 0.6065

Exercise A.7.4

1 **a** 5 **b** 2 **c** 2 **d** 1 **e** 2 **f** 1

2 **a** $\log a = \log b + \log c$ **b** $\log a = 2\log b + \log c$
c $\log a = -2\log c$ **d** $\log a = \log b + 0.5\log c$
e $\log a = 3\log b + 4\log c$ **f** $\log a = 2\log b - 0.5\log c$

3 **a** 0.18 **b** 0.045 **c** –0.09

4 **a** $x = yz$ **b** $y = x^2$ **c** $y = \frac{x+1}{x}$
d $x = 2^{y+1}$ **e** $y = \sqrt{x}$ **f** $y^2 = (x+1)^3$

5 **a** $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\frac{17}{15}$ **d** $\frac{3}{2}$ **e** $\frac{1}{3}$
f no real soln **g** 3,7 **h** $\frac{\sqrt{33}-1}{2}$ **i** 4
j $\sqrt{10}+3$ **k** $\frac{64}{63}$ **l** $\frac{2}{15}$

6 **a** $\log_3 2wx$ **b** $\log\frac{x}{47y}$ **c** $\log_a[x^2(x+1)^3]$
d $\log_a\left[\frac{(x^5)(x+1)^3}{\sqrt{2x-3}}\right]$ **e** $\log_{10}\left(\frac{y^2}{x}\right)$ **f** $\log_2\left(\frac{y}{x}\right)$

7 **a** 1 **b** –2 **c** 3 **d** 9 **e** 2 **f** 9

8 **a** 1,4 **b** $1, 3^{\pm\sqrt{3}}$ **c** $1, 4^{\sqrt[3]{4}}$ **d** $1, 5^{\pm\sqrt[4]{5}}$

9 **a** $\frac{\log 14}{\log 2} = 3.81$ **b** $\frac{\log 8}{\log 10} = 0.90$ **c** $\frac{\log 125}{\log 3} = 4.39$
d $\frac{1}{\log 2}\times\log\left(\frac{11}{3}\right) - 2 = -0.13$ **e** $\frac{\log 10 - \log 3}{4\log 3} = 0.27$
f 5.11 **g** $\frac{-\log 2}{2\log 10} = -0.15$

h 7.37 **i** 0.93 **j** no real solution

k $\frac{\log 3}{\log 2} - 2 = -0.42$ **l** $\frac{\log 1.5}{\log 3} = 0.37$

10 **a** 0.5,4 **b** 3 **c** −1,4 **d** $10, 10^{10}$ **e** 5
f 3

11 **a** $(4, \log_4 11)$ **b** 100,10 **c** 2,1

12 **a** $y = xz$ **b** $y = x^3$ **c** $x = e^{y-1}$

13 **a** $\frac{1}{e^4 - 1}$ **b** $\frac{1}{3}$ **c** $\frac{\sqrt{5} - 1}{2}$ **d** $\varnothing$

14 **a** $\ln 21 = 3.0445$ **b** $\ln 10 = 2.3026$ **c** $-\ln 7 = -1.9459$

d $\ln 2 = 0.6931$ **e** $\ln 3 = 1.0986$

f $2\ln\left(\frac{14}{9}\right) = 0.8837$ **g** $e^3 = 20.0855$

h $\frac{1}{3}e^2 = 2.4630$ **i** $\pm\sqrt{e^9} = \pm 90.0171$ **j** $\varnothing$

k $e^2 - 4 = 3.3891$ **l** $\sqrt[3]{e^9} = 20.0855$

15 **a** $0, \ln 2$ **b** $\ln 5$ **c** $\ln 2, \ln 3$ **d** 0
e $0, \ln 5$ **f** $\ln 10$

16 **a** 4.5222 **b** 0.2643 **c** 0,0.2619
d −1,0.3219 **e** −1.2925,0.6610 **f** 0,1.8928
g 0.25,2 **h** 1 **i** 121.5 **j** 2

Exercise A.8.1

1 **a** $\frac{81}{2}$ **b** $\frac{10}{13}$ **c** 5000 **d** $\frac{30}{11}$

2 $23\frac{23}{99}$

3 6667 fish. [NB: $t_{43} < 1$. If we use $n = 43$ then ans is 6660 fish]; 20 000 fish.
Overfishing means that fewer fish are caught in the long run.

4 27

5 48,12,3 or 16,12,9

6 **a** $\frac{11}{30}$ **b** $\frac{37}{99}$ **c** $\frac{191}{90}$

7 128 cm

8 $\frac{121}{9}$

9 $2 + \frac{4}{3}\sqrt{3}$

10 $\frac{1-(-t)^n}{1+t}$ $\frac{1}{1+t}$

11 $\frac{1-(-t^2)^n}{1+t^2}$ $\frac{1}{1+t^2}$

12 $^9/_5$ u^2.

Exercise A.9.1

1 a i 2 ii -3 iii 6 iv 0 v $\frac{3}{2}$ vi $\frac{1}{3}$

b i 2 ii $\sqrt{2}$ iii -5 iv $-\frac{2}{5}$ v $\frac{1}{2}$ vi -1

c i $2-2i$ ii $-3-\sqrt{2}i$ iii $6+5i$

iv $\frac{2}{5}i$ v $\frac{3}{2}-\frac{1}{2}i$ vi $\frac{1}{3}+i$

2 a $7+i$ b $1-3i$ c $15-8i$
d $-1-8i$ e $10+11i$ f $-2+3i$

3 a $-1+3i$ b $5-i$ c $-4+3i$
d $6i$ e $-4+7i$ f $-2+3i$

4 a $\frac{1}{2}(1+i)$ b $-\frac{1}{2}(5+i)$ c $-1-2i$

d $\frac{1}{2}i$ e 1 f $-\frac{1}{13}(5+i)$

5 a $14+8i$ b $-2-2i$ c $-2\sqrt{2}-i$

d $\frac{1}{5}(2+i)$ e $2-i$ f $\frac{1}{5}(1+3i)$

6 a $\frac{1}{2}$ b $\frac{1}{2}(3+\sqrt{2})$ c $3+\sqrt{2}$

7 a $x = 4, y = -\frac{1}{2}$ b $x = -5, y = 12$ c $x = 0, y = 5$

8 a i $1, i, -1, -i, 1, i$ ii $-i, -1, i, 1, -i$

b i -1 ii $-i$ iii -1 iv -1

9 $x = -\frac{120}{29}, y = \frac{39}{29}$

12 a $x = 0$ or $y = 0$ or both b $x^2-y^2 = 1$

13 a $3-i$ b $2-i$

14 a $4i$ b -4 c $-i$

15 a $x = 13, y = 4$ b $x = 4, y = \frac{4}{3}$

16 1

17 $-\frac{1}{3}(1+2\sqrt{2}i)$

18 $(u, v) = \left(\frac{1}{2}(\sqrt{2}+2), \frac{1}{2}\sqrt{2}\right), \left(\frac{1}{2}(2-\sqrt{2}), -\frac{1}{2}\sqrt{2}\right)$

19 a $-\frac{7}{2}$ b $-\frac{1}{5}$

21 $\pm\frac{\sqrt{2}(1+i)}{2}$

22 a $\cos(\theta+\alpha)+i\sin(\theta+\alpha)$ b $\cos(\theta-\alpha)+i\sin(\theta-\alpha)$

c $r_1r_2(\cos(\theta+\alpha)+i\sin(\theta+\alpha))$

d $x^2-2x\cos(\theta)+1$ e $\text{x}^2+2\text{x}\sin(\alpha)+1$

24 a $3+i$ b 325 c $(x^2+y^2)^2$

25 $z = 4, b = -4$

26 a $\cos(\theta)+i\sin(\theta)$ b $\cos(4\theta)+i\sin(4\theta)$

27 a $\begin{bmatrix}-\alpha^2 & 0\\ 0 & -\beta^2\end{bmatrix}$ b $\begin{bmatrix}\alpha^4 & 0\\ 0 & \beta^4\end{bmatrix}$

c $\begin{bmatrix}-\frac{i}{\alpha} & 0\\ 0 & \frac{i}{\beta}\end{bmatrix}$ d $\begin{bmatrix}\alpha^{4n} & 0\\ 0 & \beta^{4n}\end{bmatrix}$

28 a $-\sin(\theta)+i\cos(\theta)$ d $\cos(\theta)-i\sin(\theta)$

Exercise A.9.2

1. Show the following complex numbers on an Argand diagram:

g $\frac{1}{2i}$ h $\frac{2}{1+i}$

3. If $z_1 = 1+2i$ and $z_2 = 1+i$, show each of the following on an Argand diagram:

g $\frac{z_1}{z_2}$ h $\frac{z_2}{z_1}$

4. Find the modulus and argument of:

d $3i$ e $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$ f $\frac{1}{\sqrt{2}}(i+1)$

g 6 h $\left(1-\frac{1}{2}i\right)^2$

14. Determine the modulus and argument of each of the complex numbers:

a $3-4i$ b $\dfrac{2}{1+i}$ c $\dfrac{1-i}{1+i}$

15. If $z = 1+i$ find $Arg(z)$. hence, find $Arg\left(\dfrac{1}{z^4}\right)$.

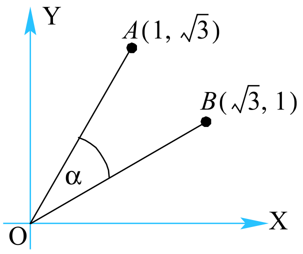
16. Determine the modulus and argument of each of the complex numbers:

a $\cos\theta + i\sin\theta$ b $\sin\theta + i\cos\theta$ c $\cos\theta - i\sin\theta$

17. Find the modulus and argument of:

a $1 + i\tan\alpha$ b $\tan\alpha - i$ c $1 + \cos\theta + i\sin\theta$

18. i Express $\dfrac{1+\sqrt{3}i}{\sqrt{3}+i}$ in the form $u+vi$.



ii Let α be the angle as shown in the diagram. Use part i to find α, clearly explaining your reason(s).

Hence, find $Arg(z)$ where $z = \left(\dfrac{1+\sqrt{3}i}{\sqrt{3}+i}\right)^7$.

19. Find:

i the modulus

ii the principal argument of the complex number $1-\cos\theta - i\sin\theta$.

On an Argand diagram, for the case $0<\theta<\pi$, interpret geometrically the relationship:

$$1-\cos\theta - i\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\left(\cos\left(\frac{\theta-\pi}{2}\right)+i\sin\left(\frac{\theta-\pi}{2}\right)\right).$$

20. If $z = \cos\theta + i\sin\theta$, prove:

a $\dfrac{2}{1+z} = 1 - i\tan\left(\dfrac{\theta}{2}\right)$.

b $\dfrac{1+z}{1-z} = i\cot\left(\dfrac{\theta}{2}\right)$.

Exercise A.9.3

1 a $\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$ b $\sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right)$ c $\sqrt{2}\text{cis}\left(-\frac{3\pi}{4}\right)$

2 a $2\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$ b $2\text{cis}\left(\frac{\pi}{6}\right)$ c $4\sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)$

d $5\text{cis}(53°7')$ e $\sqrt{5}\text{cis}(153°26')$ f $\sqrt{13}\text{cis}(-123°41')$

g $2\text{cis}\left(\frac{5\pi}{6}\right)$ h $\text{cis}\left(-\frac{\pi}{3}\right)$ i $\sqrt{10}\text{cis}(-18°26')$

3 a $2i$ b $\frac{3\sqrt{3}}{2}+\frac{3}{2}i$ c $1-i$

d $-5i$ e $-4+4\sqrt{3}i$ f $\frac{1}{6}(\sqrt{2}+\sqrt{6}i)$

4 a $\sqrt{\frac{5}{3}}$ b 1 c 0

5 a $1-\sqrt{3}i$ b $1-i$ c $(1-\sqrt{3})+(1+\sqrt{3})i$

7 a $\sqrt{2}$ b 2 c $2\sqrt{2}$

d $\frac{\pi}{4}$ e $\frac{2\pi}{3}$ f $\frac{11\pi}{12}$

Exercise A.9.4

1 a $-4(1+i)$ b -4 c $-16+16i$

d $-8-8\sqrt{3}i$ e $-16\sqrt{3}-16i$ f $-117-44i$

2 a $\frac{1}{8}(-1+i)$ b $-\frac{1}{4}$ c $-\frac{1}{32}(1+i)$

d $\frac{1}{32}(-1+\sqrt{3}i)$ e $\frac{1}{64}(-\sqrt{3}+i)$ f $\frac{1}{15625}(-117+44i)$

3 a $-8i$ b $\frac{81}{2}(-1+\sqrt{3}i)$ c $\frac{1}{2}i$

d $-\frac{1}{125}i$ e $-\frac{1}{16}(1+\sqrt{3}i)$ f $-\frac{2}{81}(1+\sqrt{3}i)$

4 a $128(1-i)$ b $4\sqrt{3}-4i$ c $-32i$

d 256 e $\frac{11753}{625}-\frac{10296}{625}i$ f $-2i$

5 b $i-1$ ii -1 iii i

6 a $-i$ b $6\sqrt{2}(1+i)$ c $-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}}i$

7 a $\frac{\sqrt{2}}{2}(1+i)$; $\frac{1}{2}(1+\sqrt{3}i)$ $\frac{\sqrt{2}}{4}((1-\sqrt{3})+(1+\sqrt{3})i)$

b i $\frac{\sqrt{2}}{4}(1+\sqrt{3})$ ii $\frac{\sqrt{2}}{4}(1-\sqrt{3})$

c $3(\mathrm{cis}(-\theta))^3$

16 $(\cos 2\theta + \cos 2\alpha)(\cos(\theta-\alpha) - i\sin(\theta-\alpha))$ [or $2\cos(\alpha-\theta)$]

18 a $\mathrm{cosec}\,\theta$ b $\theta - \frac{\pi}{2}$

Exercise A.9.5

3. Middle C.

4. $V_{A+B} = 2\sqrt{2}\cos\left(2x + \frac{\pi}{4}\right)$

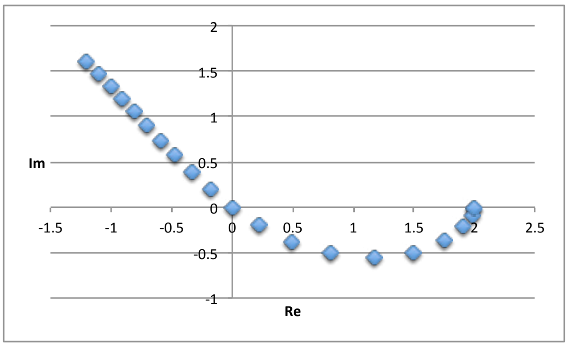
5. $W_{A+B} = 5\cos\left(x + \cos^{-1}\left(\frac{4}{5}\right)\right)$

6. $P_{\text{foxes}} + P_{\text{rabbits}} = 4.414\cos(x + 0.5363) + 10$

7. 18 patients (18.074).

8. a $I_{\text{Property}} = 2\cos(2t+3) + 20$ and $I_{\text{Shares}} = 3\cos(2t+1) + 12$ c $I_{\text{Total}} = 2.83\cos(2t + 1.698) + 32$

9.



10. a $S_{\text{Icecream}} = 4\cos(0.52t) + 17$ and $S_{\text{Chocolate}} = 5\cos(0.52t + 2.7) + 12$

c $S_{\text{Total}} = 2.20\cos(0.52t + 1.81) + 29$

Exercise A.10.1

1. $\begin{bmatrix} 67 & 73 & 83 & 59 & 92 \\ 72 & 81 & 80 & 65 & 88 \end{bmatrix}$

   2 by 5

2. Only C & E

3. a $\begin{bmatrix} 18 & 7 \\ 3 & 11 \end{bmatrix}$ b $\begin{bmatrix} 4 & 14 \\ 8 & 10 \end{bmatrix}$ c $\begin{bmatrix} 17 & 18 \\ 11 & 8 \end{bmatrix}$ d $\begin{bmatrix} 1 & 0 \\ -2 & -8 \end{bmatrix}$

   e $\begin{bmatrix} -3 & 5 \\ -7 & 6 \end{bmatrix}$ f $\begin{bmatrix} -7 & -7 \\ -19 & 16 \end{bmatrix}$

4. a $\begin{bmatrix} -7 & -1 \\ 6 & 5 \\ -1 & -2 \end{bmatrix}$ b $\begin{bmatrix} -3 & 1 \\ 0 & -3 \\ -5 & 6 \end{bmatrix}$ c $\begin{bmatrix} -16 & -3 \\ 15 & 14 \\ 0 & -8 \end{bmatrix}$ d $\begin{bmatrix} -11 & 2 \\ 3 & -5 \\ -13 & 14 \end{bmatrix}$

   e $\begin{bmatrix} 4 & -3 \\ 3 & 10 \\ 12 & -16 \end{bmatrix}$

5. 2012: $\begin{bmatrix} 11 & 44 & 50 & 45 \\ 67 & 81 & 35 & 93 \\ 13 & 8 & 53 & 59 \\ 55 & 74 & 5 & 76 \\ 21 & 79 & 61 & 99 \end{bmatrix}$ 2013: $\begin{bmatrix} 11 & 59 & 8 & 31 \\ 64 & 47 & 39 & 37 \\ 64 & 13 & 87 & 24 \\ 95 & 22 & 78 & 2 \\ 57 & 21 & 21 & 17 \end{bmatrix}$ 2014: $\begin{bmatrix} 66 & 33 & 71 & 1 \\ 56 & 47 & 87 & 45 \\ 93 & 15 & 30 & 76 \\ 64 & 78 & 6 & 60 \\ 92 & 20 & 51 & 4 \end{bmatrix}$

   The sum of these three matrices represents the sum of the purchases over the three years.

   $$\begin{bmatrix} 88 & 136 & 129 & 77 \\ 187 & 175 & 161 & 175 \\ 170 & 36 & 170 & 159 \\ 214 & 174 & 89 & 138 \\ 170 & 120 & 133 & 120 \end{bmatrix}$$

Exercise A.10.2

1. $A \times C$, $B \times C$, $C \times B$, $B \times A$.

2. a $\begin{bmatrix} 9 & 9 \\ 6 & 7 \end{bmatrix}$ b $\begin{bmatrix} 3 & 12 \\ 10 & 22 \end{bmatrix}$ c $\begin{bmatrix} -16 & 16 \\ 15 & -6 \end{bmatrix}$

   d $\begin{bmatrix} 34 & 41 \\ 11 & 4 \end{bmatrix}$ e $\begin{bmatrix} -17 & 16 \\ -22 & 14 \end{bmatrix}$

3. a $\begin{bmatrix} 14 & 9 & 4 \\ 15 & 12 & 6 \\ 12 & 3 & 0 \end{bmatrix}$ b $\begin{bmatrix} 20 & 12 \\ 17 & 3 \end{bmatrix}$ c $\begin{bmatrix} 11 & -26 & 7 \\ -7 & 28 & -11 \\ 16 & -6 & -8 \end{bmatrix}$

d $\begin{bmatrix} -4 & 4 \\ 8 & 10 \end{bmatrix}$ e $\begin{bmatrix} 9 & 7 & 2 \\ 9 & 5 & 1 \\ 6 & -2 & -2 \end{bmatrix}$

4. $\begin{bmatrix} 38 & 67 & 45 \\ 42 & 56 & 27 \\ 84 & 102 & 74 \end{bmatrix} \times \begin{bmatrix} 0.75 \\ 0.95 \\ 0.55 \end{bmatrix} = \begin{bmatrix} 116.9 \\ 99.55 \\ 200.6 \end{bmatrix}$

5. $\begin{bmatrix} 78 & 127 & 1.2 \\ 85 & 124 & 1.1 \\ 98 & 125 & 1.3 \\ 79 & 126 & 1.2 \\ 76 & 123 & 1.3 \\ 82 & 120 & 1.3 \\ 86 & 122 & 1.2 \\ 88 & 127 & 1.3 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 78 & 127 & 1.2 \\ 88 & 124 & 1.1 \\ 101 & 125 & 1.3 \\ 82 & 126 & 1.2 \\ 79 & 123 & 1.3 \\ 85 & 120 & 1.3 \\ 89 & 122 & 1.2 \\ 91 & 127 & 1.3 \end{bmatrix}$

6. a $AB = \begin{bmatrix} 4 & -11 \\ 8 & -25 \end{bmatrix} \neq BA = \begin{bmatrix} -9 & 12 \\ 10 & -12 \end{bmatrix}$ non-commutative

b $AC = \begin{bmatrix} 2 & -7 \\ 6 & -15 \end{bmatrix} \neq CA = \begin{bmatrix} -1 & 0 \\ 9 & -12 \end{bmatrix}$ non-commutative

c $(AB)C = \left(\begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} \times \begin{bmatrix} 0 & 3 \\ 2 & -4 \end{bmatrix} \right) \times \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 4 & -11 \\ 8 & -25 \end{bmatrix} \times \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} -8 & 37 \\ -16 & 83 \end{bmatrix}$

$A(BC) = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} \times \left(\begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} \times \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix} \right) = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} \times \begin{bmatrix} 0 & -9 \\ -4 & 14 \end{bmatrix} = \begin{bmatrix} -8 & 37 \\ -16 & 83 \end{bmatrix}$

calculation is associative

d both are: $\begin{bmatrix} 18 & -45 \\ -20 & 46 \end{bmatrix}$

e $A(B+C) = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} \times \left(\begin{bmatrix} 0 & 3 \\ 2 & -4 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix} \right) = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} \times \begin{bmatrix} -2 & 4 \\ 2 & -7 \end{bmatrix} = \begin{bmatrix} 6 & -18 \\ 14 & -40 \end{bmatrix}$

$AB + AC = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} \times \begin{bmatrix} 0 & 3 \\ 2 & -4 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} \times \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 4 & -11 \\ 8 & -25 \end{bmatrix} + \begin{bmatrix} 2 & -7 \\ 6 & -15 \end{bmatrix} = \begin{bmatrix} 6 & -18 \\ 14 & -40 \end{bmatrix}$

Demonstrates the distributive property.

f Both are: $\begin{bmatrix} -9 & 3 \\ 6 & 2 \end{bmatrix}$

Exercise A.10.3

1\. a $\begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix}$ b $\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$ c undefined

d $\begin{bmatrix} 1 & 0 \\ \frac{1}{3} & \frac{1}{6} \end{bmatrix}$ e $\begin{bmatrix} 1.75 & 1 \\ 1.5 & 1 \end{bmatrix}$ f $\begin{bmatrix} -0.5 & -0.5 \\ -0.75 & -0.5 \end{bmatrix}$

Exercise A.10.4

1\. a $x = 0, y = 0$ & $z = 1$ b $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

c $\begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}$ d $\begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$ e $\begin{bmatrix} -1.5 \\ 1.5 \\ 2.5 \end{bmatrix}$

2\. a $a = 1, b = 2, c = 3$ & $d = 0$ b $\begin{bmatrix} 4 \\ 2 \\ 3 \\ 1 \end{bmatrix}$ c $\begin{bmatrix} -2 \\ 1 \\ 3 \\ -4 \end{bmatrix}$

d $\begin{bmatrix} 0 \\ -5 \\ 7 \\ 1 \end{bmatrix}$ e $\begin{bmatrix} 0.5 \\ -1.5 \\ 2.5 \\ 0 \end{bmatrix}$

3\. $\begin{bmatrix} \text{A-set} \\ \text{B-lap} \\ \text{C-disc} \\ \text{D-cell} \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 2 \\ 5 \end{bmatrix}$

4\. $\begin{bmatrix} \text{CD1} \\ \text{CD2} \\ \text{CD3} \\ \text{CD4} \end{bmatrix} = \begin{bmatrix} 35 \\ 19 \\ 23 \\ 17 \end{bmatrix}$

5\. This information is insufficient as order 4 is the sum of orders 2 & 3.

6\. The bottom equation is $2g = 2$ giving $g = 1$. The equation above is now $2f + 1 = 3$ so $f = 1$. The process can continue up the system.

Each row will yield a new variable's value. This is known as 'back substitution'.

The form of this particular matrix is called 'row echelon'.

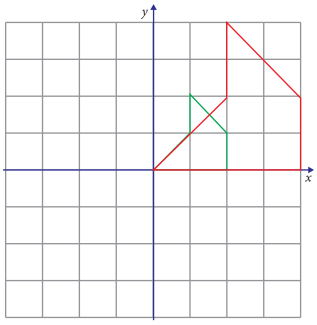
If you get data in this form, you might be better to use this method.

7. $$\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 2 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

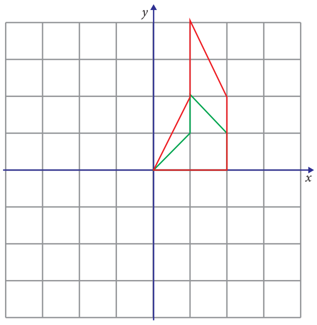
8. Row 3 is the sum of rows 1 & 2 on the LHS matrix so it is singular. This is not the case on the RHS. No solution.

Exercise A.10.5

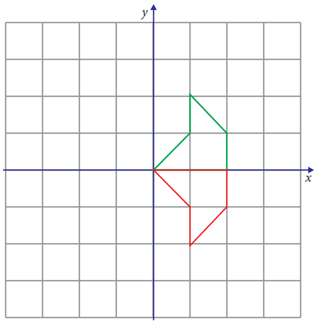
1. a



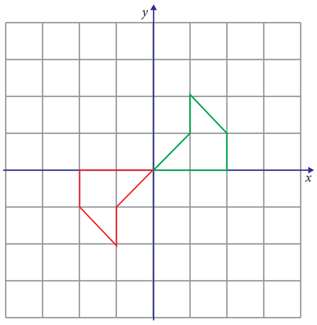
b



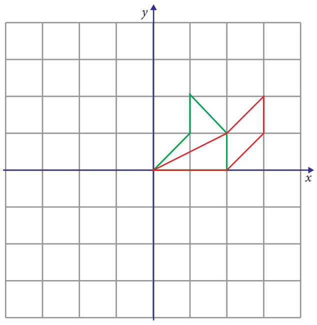
c



d



e



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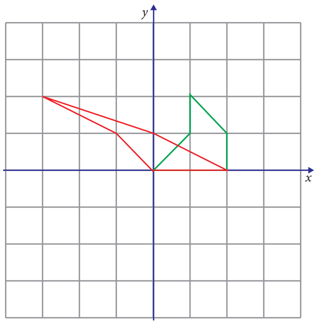

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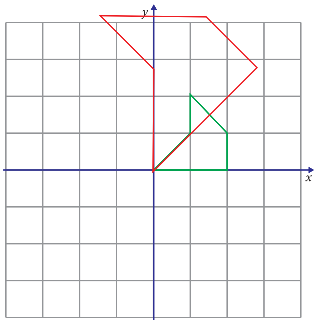

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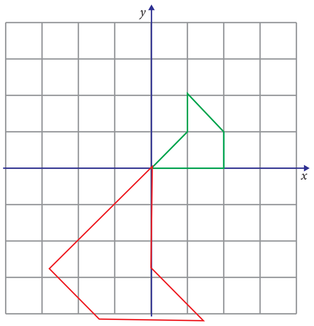


2. a $\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$ b $\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$ c $\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$

d $\begin{bmatrix}-2 & 0\\0 & 2\end{bmatrix}$ e $\begin{bmatrix}0 & -2\\-2 & 0\end{bmatrix}$ f $\begin{bmatrix}0 & 2\\2 & 0\end{bmatrix}$

Exercise A.10.6

1. a $4,\begin{bmatrix}5\\2\end{bmatrix}$ or $-1,\begin{bmatrix}0\\1\end{bmatrix}$ b $\sqrt{2},\begin{bmatrix}1\\\sqrt{2}\end{bmatrix}$ or $-\sqrt{2},\begin{bmatrix}1\\-\sqrt{2}\end{bmatrix}$

c $0,\begin{bmatrix}1\\-1\end{bmatrix}$ or $3,\begin{bmatrix}1\\2\end{bmatrix}$ d $2,\begin{bmatrix}1\\0\end{bmatrix}$ or $5,\begin{bmatrix}0\\1\end{bmatrix}$

e $1,\begin{bmatrix}1\\0\end{bmatrix}$

2. a k^2-7k+6 b 1,6 c $\begin{bmatrix}1\\-3\end{bmatrix},\begin{bmatrix}1\\2\end{bmatrix}$

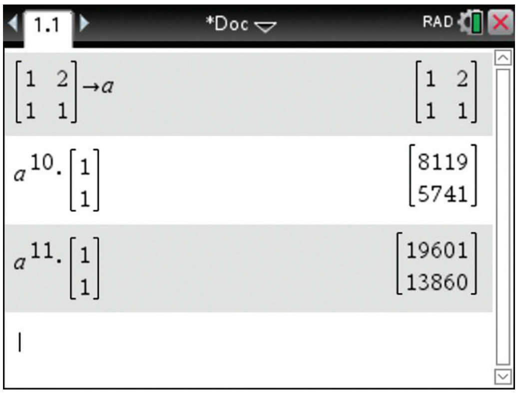
3. a $p_{n+1}=p_n+2q_n$

$q_{n+1}=p_n+q_n$

b $\begin{bmatrix}p_{n+1}\\q_{n+1}\end{bmatrix}=\begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}\times\begin{bmatrix}p_n\\q_n\end{bmatrix}$

c k^2-2k-1 d $1\pm\sqrt{2}$

e



$\begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}\to a$ $\begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}$

$a^{10}\cdot\begin{bmatrix}1\\1\end{bmatrix}$ $\begin{bmatrix}8119\\5741\end{bmatrix}$

$a^{11}\cdot\begin{bmatrix}1\\1\end{bmatrix}$ $\begin{bmatrix}19601\\13860\end{bmatrix}$

f The ratio of the second of the two examples is: 19601/13860 ≈ 1.414 ≈√2

4. b $$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & -B \\ -A & -1 \end{bmatrix} \times \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

c $$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & -0.25 \\ -0.2 & 1 \end{bmatrix} \times \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$ so that $$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 & -0.25 \\ -0.2 & 1 \end{bmatrix} \times \begin{bmatrix} 55 \\ 50 \end{bmatrix} = \begin{bmatrix} 42.5 \\ 39 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -0.25 \\ -0.2 & 1 \end{bmatrix}^{11} \times \begin{bmatrix} 55 \\ 50 \end{bmatrix} \approx \begin{bmatrix} -0.7 \\ 6.8 \end{bmatrix}$$ so Force Y has won after 11 hours.

Though if Force X has any sense, it will have surrendered long before this!

Exercise B.6.1

3 All of the following functions are mappings of $\mathbb{R}\mapsto\mathbb{R}$ unless otherwise stated.

a Determine the composite functions $(f\circ g)(x)$ and $(g\circ f)(x)$, if they exist.

b For the composite functions in part a that do exist, find their range.

viii $f(x) = x-4, g(x) = |x|$

ix $f(x) = x^3-2, g(x) = |x+2|$

x $f(x) = \sqrt{4-x}, x \le 4, g(x) = x^2$

xi $f(x) = \frac{x}{x+1}, x \ne -1, g(x) = x^2$

xii $f(x) = x^2+x+1, g(x) = |x|$

xiii $f(x) = 2^x, g(x) = x^2$

xiv $f(x) = \frac{1}{x+1}, x \ne -1, g(x) = x-1$

xv $f(x) = \frac{2}{\sqrt{x-1}}, x > 1, g(x) = x^2+1$

xvi $f(x) = 4^x, g(x) = \sqrt{x}$

13 Find $(h\circ f)(x)$, given that $h(x) = \begin{cases} x^2+4, x \ge 1 \\ 4-x, \ \ x < 1 \end{cases}$ and $f: x \mapsto x-1, x \in \mathbb{R}$.

Sketch the graph of $(h\circ f)(x)$ and use it to find its range.

14 a Given three functions, *f*, *g* and *h*, when would $h\circ g\circ f$ exist?

b If $f: x \mapsto x+1, x \in \mathbb{R}$, $g: x \mapsto x^2, x \in \mathbb{R}$ and $h: x \mapsto 4x, x \in \mathbb{R}$, find $(h\circ g\circ f)(x)$.

15 Given the functions $f(x) = e^{2x-1}$ and $g(x) = \frac{1}{2}(\ln x+1)$ find, where they exist:

a $(f\circ g)$ b $(g\circ f)$ c $(f\circ f)$

In each case find the range of the composite function.

16 Given that $h(x) = \log_{10}(4x-1), x > \frac{1}{4}$ and $k(x) = 4x - 1, x \in]-\infty,\infty[$, find, where they exist :

a $(h\circ k)$ b $(k\circ h)$.

17 Given the functions $f(x) = \sqrt{x^2-9}, x \in \mathbf{S}$ and $g(x) = |x|-3, x \in \mathbf{T}$, find the largest positive subsets of $\mathbb{R}$ so that:

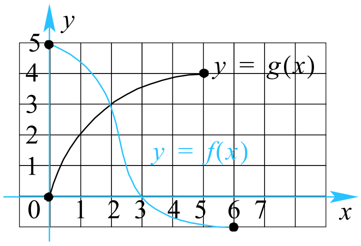
a $g\circ f$ exists b $f\circ g$ exists.

18 For each of the following functions:

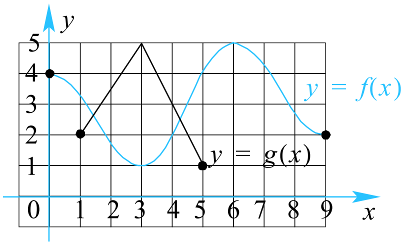
a determine if $f\circ g$ exists and sketch the graph of $f\circ g$ when it exists.

b determine if $g\circ f$ exists and sketch the graph of $g\circ f$ when it exists.

i



ii



19 Given the functions $f: \mathrm{S} \to \mathbb{R}$ where $f(x) = e^{x+1}$ and $g: \mathrm{S} \to \mathbb{R}$ where $g(x) = \ln 2x$ where $\mathrm{S} =]0, \infty[$.

a State the domain and range of both f and g.

b Giving reasons, show that $g \circ f$ exists but $f \circ g$ does not exist.

c Fully define $g \circ f$, sketch its graph and state its range.

20 The functions *f* and *g* are given by $f(x) = \begin{cases} \sqrt{x-1} & \text{if } x \geq 1 \\ x-1 & \text{if } 0 < x < 1 \end{cases}$ and $g(x) = x^2 + 1$.

a Show that $f \circ g$ is defined. b Find $(f \circ g)(x)$ and determine its range.

21 Let $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ where $f(x) = \begin{cases} \dfrac{1}{x^2}, & 0 < x \leq 1 \\ \dfrac{1}{\sqrt{x}}, & x > 1 \end{cases}$.

a Sketch the graph of *f*.

b Define the composition $f \circ f$, justifying its existence.

c Sketch the graph of $f \circ f$, giving its range.

22 Consider the functions $f :]1,\infty[\to \mathbb{R}$ where $f(x) = \sqrt{x}$ and $g : \mathbb{R} \backslash \{0\} \to \mathbb{R}$ where $g(x) = x^2$.

a Sketch the graphs of *f* and *g* on the same set of axes.

b Prove that $g \circ f$ exists and find its rule.

c Prove that $f \circ g$ cannot exist.

d If a new function $g^* : S \to \mathbb{R}$ where $g^*(x) = g(x)$ is now defined, find the largest positive subset of $\mathbb{R}$ so that $f \circ g^*$ does exist. Find $f \circ g^*$, sketch its graph and determine its range.

23 Given that $f(x) = \dfrac{ax-b}{cx-a}$, show that $f \circ f$ exists and find its rule.

24 a Sketch the graphs of $f(x) = \dfrac{1}{a}x^2$ and $g(x) = \sqrt{2a^2 - x^2}$, where $a > 0$.

b Show that $f \circ g$ exists, find its rule and state its domain.

c Let S be the largest subset of $\mathbb{R}$ so that $g \circ f$ exists.

i Find S.

ii Fully define $g \circ f$, sketch its graph and find its range.

Exercise B.6.2

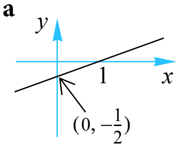
1 a $\frac{1}{2}(x-1), x \in \mathbb{R}$ b $\sqrt[3]{x}, x \in \mathbb{R}$

c $3(x+3), x \in \mathbb{R}$ d $\frac{5}{2}(x-2), x \in \mathbb{R}$

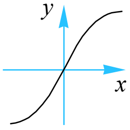
e $x^2 - 1, x > 0$ f $(x-1)^2, x \geq 1$

g $\frac{1}{x} - 1, x > 0$ h $\dfrac{1}{(x+1)^2}, x > -1$

2 a



b



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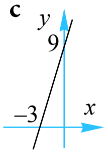

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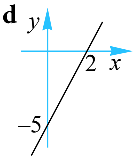
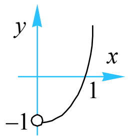
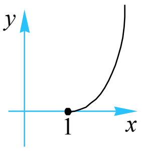


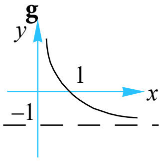
e



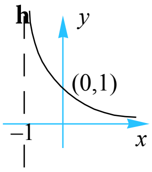
f



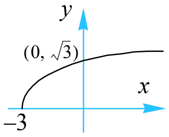
g



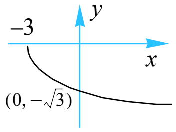
h



3 a $\sqrt{x+3}, x \geq -3$

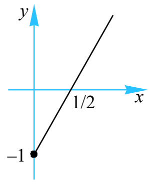


b $-\sqrt{x+3}, x \geq -3$



4 $\frac{\pm|x|}{\sqrt{1-x^2}}, -1 < x < 1$

5 a



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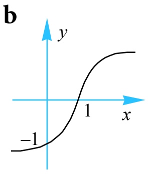

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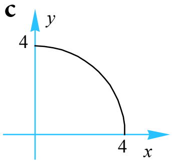

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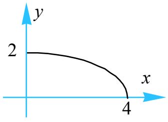

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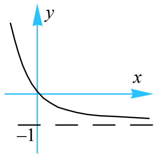

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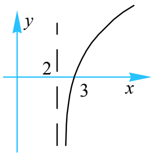

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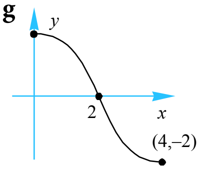

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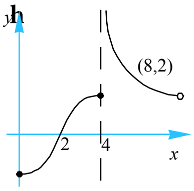


6

a $f^{-1}(x) = \log_3(x-1), x > 1$

b $f^{-1}(x) = \log_2(x+5), x > -5$

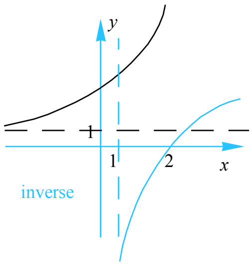
c $f^{-1}(x) = \frac{1}{2}(\log_3 x - 1), x > 0$

d $g^{-1}(x) = 1 + \log_{10}(3-x), x < 3$

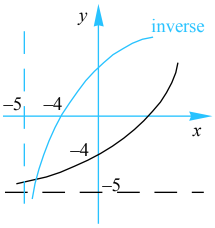
e $h^{-1}(x) = \log_3\left(1 + \frac{2}{x}\right), x \in \mathbb{R}\backslash[-2,0]$

f $g^{-1}(x) = \log_2\left(\frac{1}{x+1}\right), x > -1$

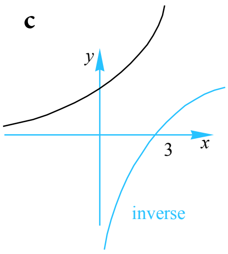
7 a



b



c



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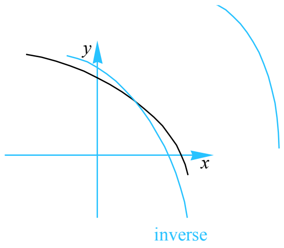

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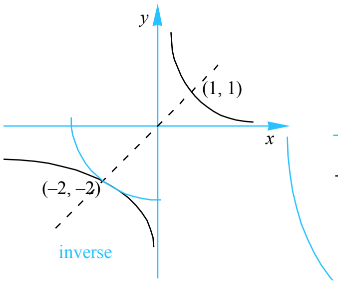

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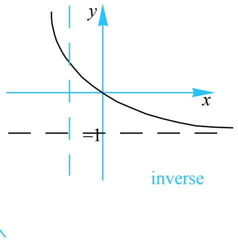


8 a $f^{-1}(x) = 2^x - 1, x \in \mathbb{R}$

b $f^{-1}(x) = \frac{1}{2} \cdot 10^x, x \in \mathbb{R}$

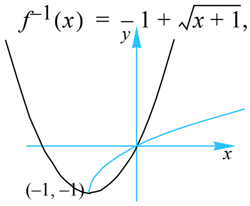
c $f^{-1}(x) = 2^{1-x}, x \in \mathbb{R}$

d $f^{-1}(x) = 3^{x+1} + 1, x \in \mathbb{R}$

e $f^{-1}(x) = 5^{x/2} + 5, x \in \mathbb{R}$

f $f^{-1}(x) = 1 - 10^{3(2-x)}, x \in \mathbb{R}$

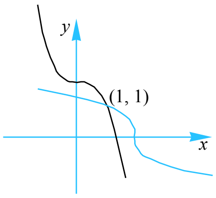
9 $f^{-1}(x) = -1 + \sqrt{x+1}, x > -1$



dom = [–1, ∞[, ran = [–1, ∞[

10 a $f^{-1}(x) = a - x$ **b** $f^{-1}(x) = \frac{2}{x-a} + a$ **c** $f^{-1}(x) = \sqrt{a^2 - x^2}$

11 $f^{-1}(x) = \sqrt[3]{2-x}$



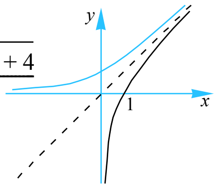
12 [2, ∞[

13 $\mathbb{R}^+ \backslash \{1.5\}$

14 a Inverse exists as *f* is one:one

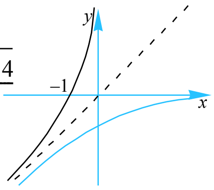
b Case 1: *S* =]0, ∞[

$g^{-1}(x) = \frac{x + \sqrt{x^2+4}}{2}$

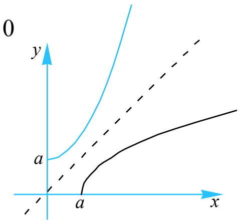


Case 2: *S* =]–∞, 0[

$g^{-1}(x) = \frac{x - \sqrt{x^2+4}}{2}$

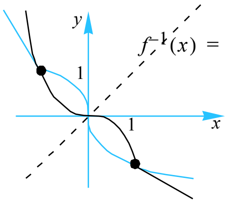


15 $f^{-1}(x) = a(x^2 + 1), x \geq 0$



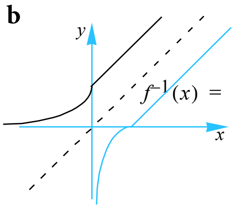
$\{x: \text{f}(x) = f^{-1}(x)\} = \varnothing$

16 a



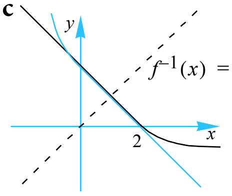
$$f^{-1}(x) = \begin{cases} -(2x+1), & x < -1 \\ -\sqrt[3]{x}, & x \geq -1 \end{cases}$$

b



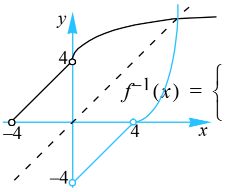
$$f^{-1}(x) = \begin{cases} \ln(x) - 1, & 0 < x \leq \text{e} \\ x - e, & x > \text{e} \end{cases}$$

c



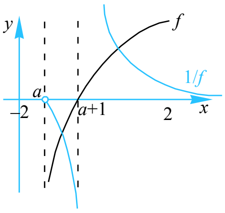
$$f^{-1}(x) = \begin{cases} 1 + e^{-x}, & x < 0 \\ 2 - x, & x \geq 0 \end{cases}$$

d

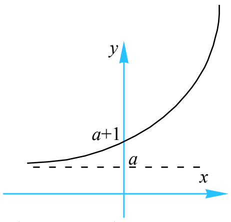


$$f^{-1}(x) = \begin{cases} (x-4)^2, & x > 4 \\ x - 4, & 0 < x < 4 \end{cases}$$

17 a



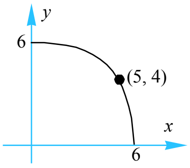
b



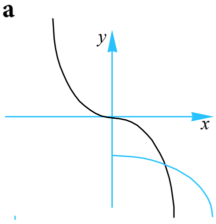
$f^{-1}\mathbb{R} \mapsto \mathbb{R}, f^{-1}(x) = a + e^{ax}$

18 *gof* exists as $r_f \subseteq d_g$.

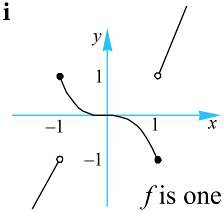
It is one:one so the inverse exists:



19 a



b i

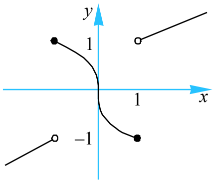


f is one:one

ii

$$f(x) = \begin{cases} \frac{1}{2}(x-1) & x < -1 \\ -\sqrt[3]{x} & -1 \leq x \leq 1 \\ \frac{1}{2}(x+1) & x > 1 \end{cases}$$

iii



iv $\{-1, 0, 1\}$

20 a i $tom(x) = e^{\sqrt{x}}, x \geq 0$ ii $mot(x) = \sqrt{e^x}, x \in \mathbb{R}$

b i $(tom)^{-1}(x) = (\ln(x))^2, x > 1$ ii $(mot)^{-1}(x) = \ln x^2, x > 0$

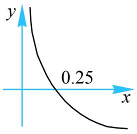
c i & ii neither exist

d Adjusting domains so that the functions in part c exist, we have:

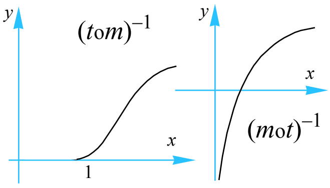
$t^{-1}om^{-1}(x) = (mot)^{-1}(x)$ and $m^{-1}ot^{-1}(x) = (tom)^{-1}(x)$

e Yes as rules of composition OK.

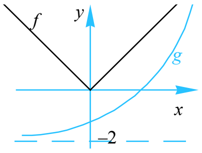
21 a 1 b



0.206



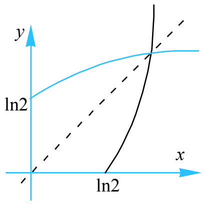
22 a



b *fog* exists but is not one:one

c i B = [ln2, ∞[

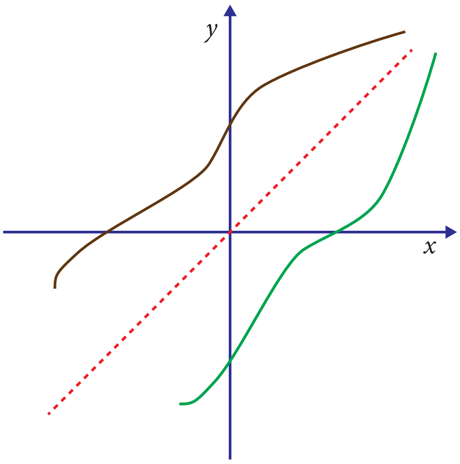
ii $(fog)^{-1}$: [0,∞[↦ ℝ where, $(fog)^{-1}(x) = \ln(x+2)$ iii



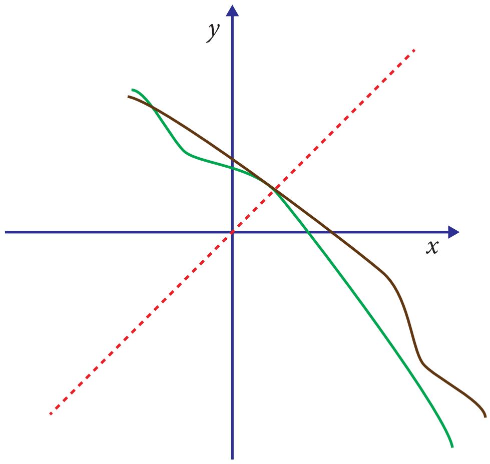
Exercise B.6.3

1. Answers not unique.

a



b



2. $f^{-1}(x)=x^2+1, x\geq 0$

3. 2

4. $f^{-1}(x)=\dfrac{\ln(x+1)}{2}, x\geq -1$

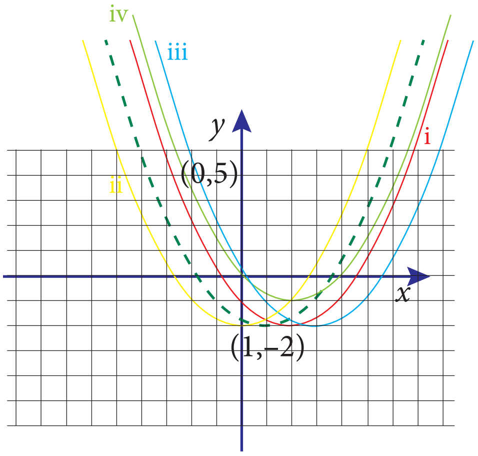
5. $f^{-1}(x)=\sqrt{\log_2(x+3)}, x\geq -2$

6. 1

7. 2

Exercise B.7.1

1. a

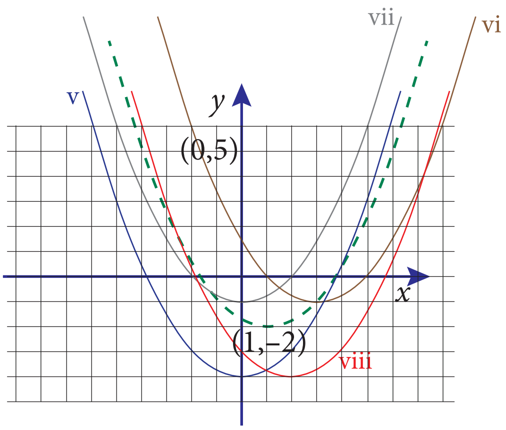


$y=f(x-1)$

$y=f(x+1)$

$y=f(x-2)$

$y=f(x-1)+1$



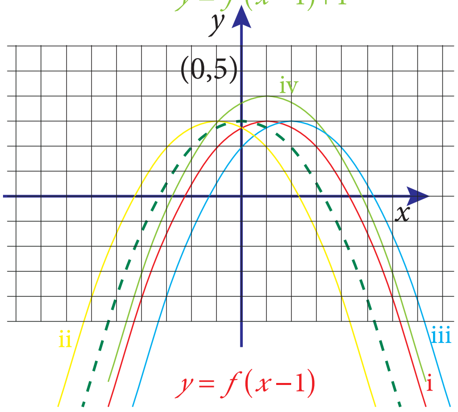
$y=f(x+1)-2$

$y=f(x-2)+1$

$y=f(x+1)+1$

$y=f(x-1)-2$

b

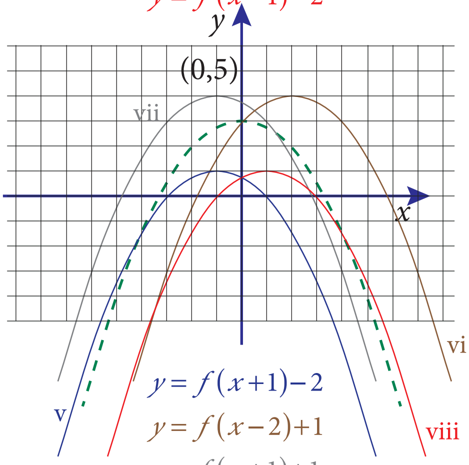


$y=f(x-1)$

$y=f(x+1)$

$y=f(x-2)$

$y=f(x-1)+1$



$y=f(x+1)-2$

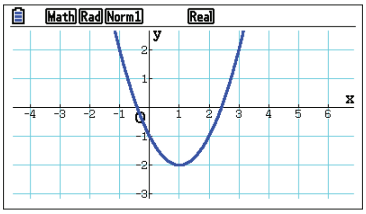
$y=f(x-2)+1$

$y=f(x+1)+1$

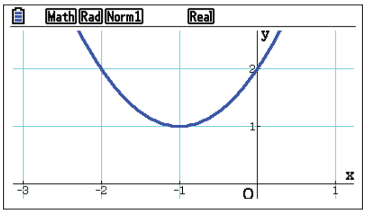
$y=f(x-1)-2$

2.

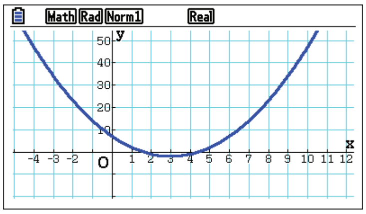
a



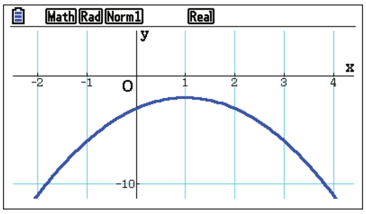
b



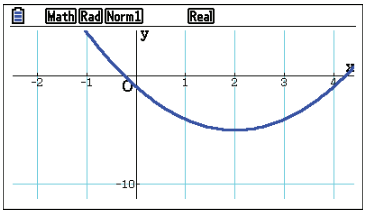
c



d



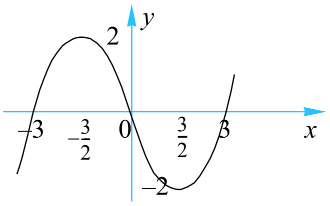
e



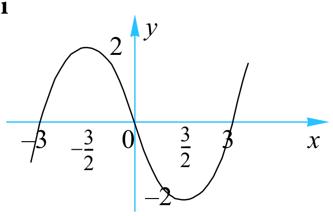
3. a $x = -2$ b $x = 4$ c $x = -1$ d $x = 12$

Exercise B.7.2

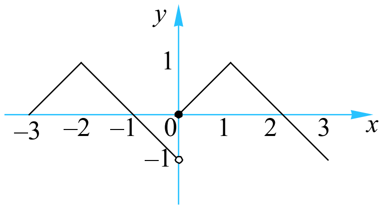
1 a i



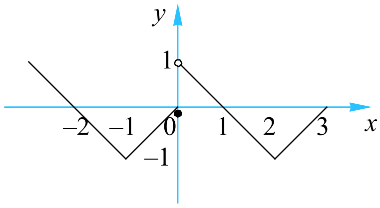
ii



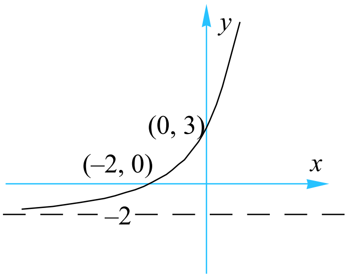
b i



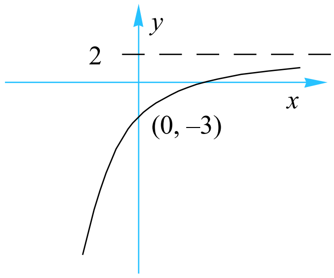
ii



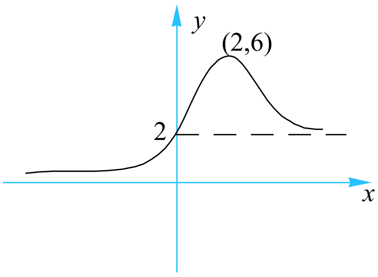
c i



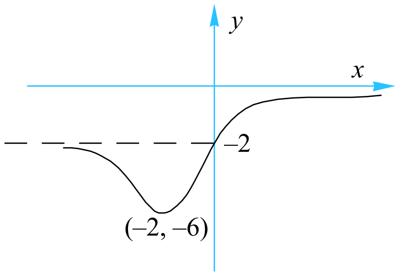
ii



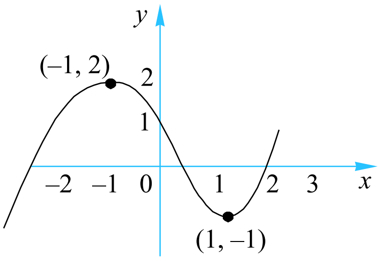
d i



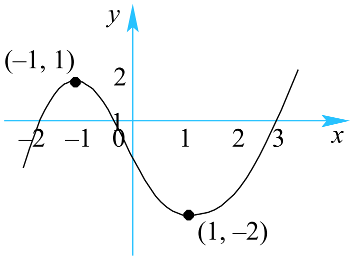
ii



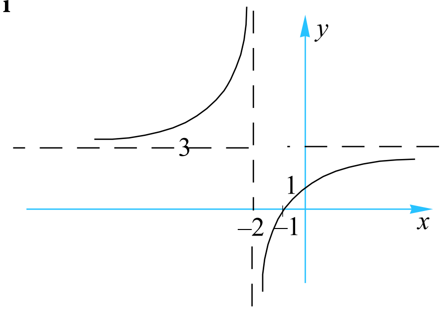
e i



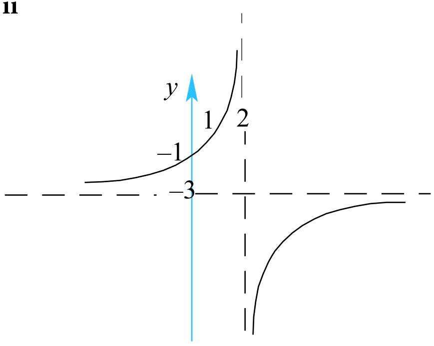
ii



f i



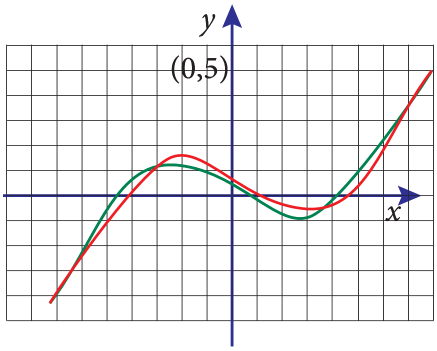
ii



2 a $y = -f(x)$ b $y = f(-x)$ c $y = f(x+1)$

d $y = f(2x)$ e $y = 2f(x)$

3.



4. $f(x) \dashrightarrow f(2x)$

5. The temperatures must be scaled using $F = 1.8C + 32$

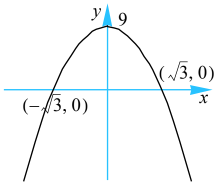
6. $f(x) \dashrightarrow -2\,f(x+2) + 4$

Adjusted reading = 1.1 × raw reading – 5

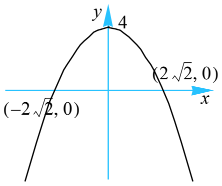
Exercise B.7.3

1

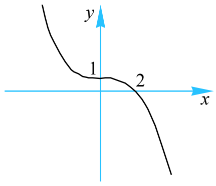
a



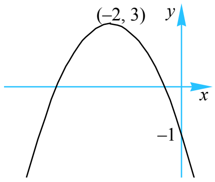
b



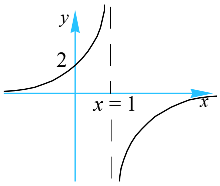
c



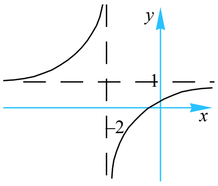
d



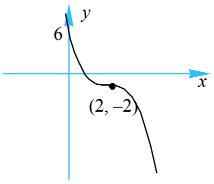
e



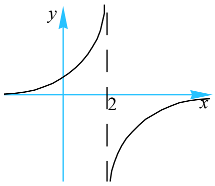
f



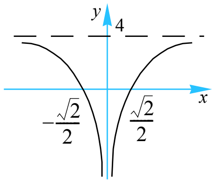
g



h



i



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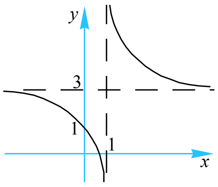

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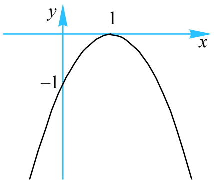

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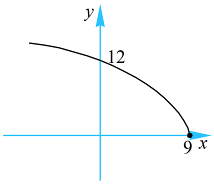
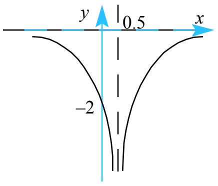
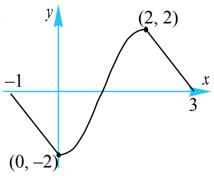


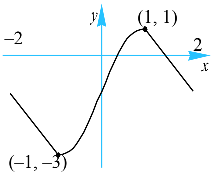
m



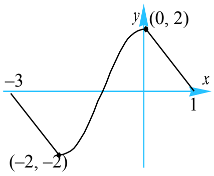
2 a



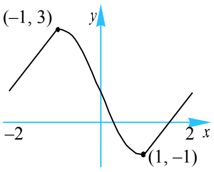
b



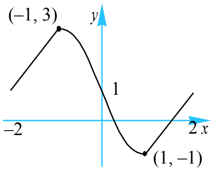
c



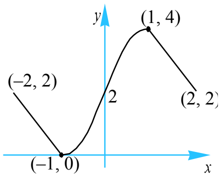
d



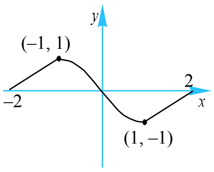
e



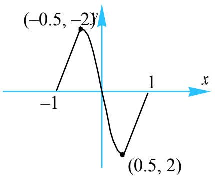
f



g



h



Exercise B.7.4

1. $p^* = \dfrac{11-0.33t}{1.1} = 10-0.3t$

2. $P^*(t) = 7\sin(1024\pi t)$

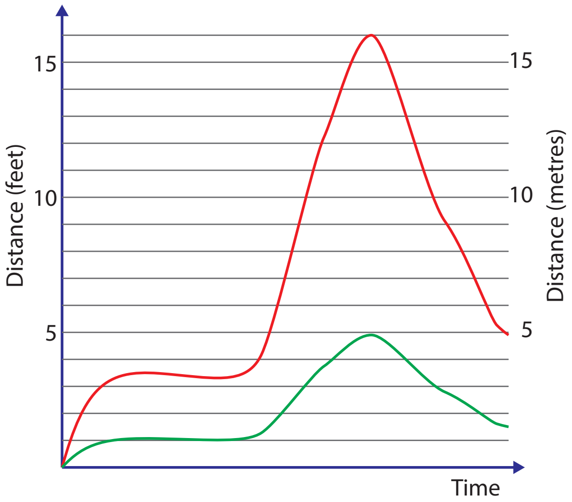
3. $R_t^* = 700e^{-2t} + 120$

4. a 5% b 26%

5. Dilation parallel to the time axis.

6. The clock is unning fast and the themometer is reading too high.

7.

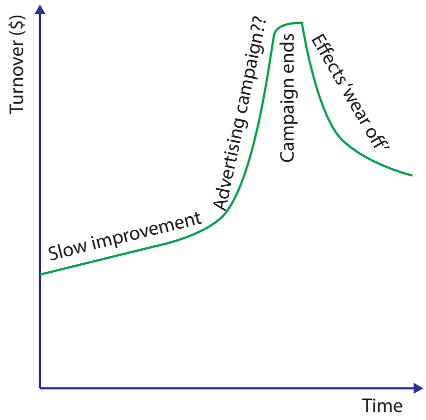


8. $M^* = \frac{t}{60} + \frac{t^2}{3600}, 0 \le t \le 120$

Exercise B.7.5

1. a ½ b ⅓ c 13 d 15

2. Many possible explanations. Example only...



3. $A = 6$, $B = 5$

4. a 0.787 b 0.951 c 1.422 d 1.896

5. a 5 b 3

6.

x	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
f(x)	1.3333	1.9069	2.4963	3.0062	3.3857	3.6378	3.7927	3.8835	3.9352	3.9642	3.9803

7. a 600 b 9 c $f(t) = \frac{600}{1+9e^{-1.098t}}, t \ge 0$ d $f(1) \approx 150$ so all is well.

8. $f(t) = \frac{1200}{1+9e^{-1.8t}}, t \ge 0$ fits the data quite well and it predicts 1 199 students after 5 years.

9. Under the existing system, the tax is £3 590. Under A, the tax is £2 692.50 (£897.50 less).

Under B, the tax is £2 960 (£630 less). Option A is better.

10. a $f(t) = \frac{5600}{1+9e^{-1.02t}}, t \ge 0$ c Cases stabilise at 5600

11. $k = 0.4$. The n value that gives $R = 75$ is ≈ 1.3 Since the test light is 50 times normal sunlight, this is 65 days.

This is well inside the requirement as normal usage is only 20 minutes sunlight per day meaning that the cloth can be expected to fade to 80% after 4 680 days.

12. $P(8) = 500$, so after the storm the population becomes 250. It is appropriate to alter the parameter C (to achieve an initial value to be 250) as this translates the graph horizontally. $C = 1.4$ does this. After a further 1.7 years the population is back to 300.

Exercise B.8.1

1. a $y=0.1x^{2}$ b $y=0.62x^{2.1}$ c $y=0.31x^{2.7}$
   d $y=0.31x^{4.1}$ e $y=0.56x^{3.9}$

2. $D=0.0358\times S^{1.7}$, 90 metres

3. $(PV)^{1.6}$ = 80 - 1.55 litres.

4. $T=35700\times D^{-2.22}$

5. $L\approx 2\times 10^{-35}T^{9}$ (very approx.)

Exercise B.8.2

1. a $y=0.5\times 1.5^{x}$ b $y=1.65\times 2.25^{x}$ c $y=4.1\times 1.9^{x}$
   d $y=2.9\times 3.97^{x}$ e $y=1.1\times 2.1^{x}$

2. a $A(t)=A_0\times e^{-1.3\times 10^{-4}t}$ b 6 500 years (very approx !)

3. a $S=2000\times 1.014^{P}$ b About 72 cents.

4. 6.16 days

5. Approx $K_p=5.063\times 1.0254^{-T}$

6. a $y=0.5\times 1.8^{x}$ b $y=2\times 1.4^{x}$ c $y=4.4\times 1.6^{-x}$

Exercise B.8.3

1. $[H]=2\times 10^{-8}\times e^{0.3365t}$

2. $C=200\times 2.6^{t}$

3. $y=x^{4.6}$

4. $y=3.7^{x}$

5.

Type	Wavelength	Frequency
Ultra-violet light	10 nm = 10^{-8} m	30 PHz = 3 × 10^{16} Hz
Visible light	1 μm = 10^{-6} m	300 THz = 3 × 10^{14} Hz
Infra-red light	100 μm = 10^{-4} m	3 THz = 3 × 10^{12} Hz
Microwaves	10 m = 10^{1} m	30 MHz = 3 × 10^{7} Hz
Radio Waves	1 km = 10^{3} m	300 kHz = 3 × 10^{5} Hz

As far as the rule is concerned, you may notice that Wavelength × Frequency = 3 × 10^{8} this is the speed of light ms^{-1}.

6. $E = 3\times T^4$. This is known as Stefan's Law.

7. Various Answers

8. $y = 2.65^x$

9. $y = 6.3^x$

10. $y = 7.2^x$

11. $y = x^{0.94}$

12. $y = x^{1.73}$

13. $y = x^{2.91}$

14. $y = 2.47^x$

Exercise C.8.1

1	**a**	120°	**b**	108°	**c**	216°	**d**	50°
2	**a**	π^c	**b**	$\frac{3\pi^c}{2}$	**c**	$\frac{7\pi^c}{9}$	**d**	$\frac{16\pi^c}{9}$
3	**a**	$\frac{\sqrt{3}}{2}$	**b**	$-\frac{1}{2}$	**c**	$-\sqrt{3}$	**d**	-2
	e	$-\frac{1}{2}$	**f**	$-\frac{\sqrt{3}}{2}$	**g**	$\frac{1}{\sqrt{3}}$	**h**	$\sqrt{3}$
	i	$-\frac{1}{\sqrt{2}}$	**j**	$-\frac{1}{\sqrt{2}}$	**k**	1	**l**	$-\sqrt{2}$
	m	$-\frac{1}{\sqrt{2}}$	**n**	$\frac{1}{\sqrt{2}}$	**o**	-1	**p**	$\sqrt{2}$
	q	0	**r**	1	**s**	0	**t**	undefined
4	**a**	0	**b**	-1	**c**	0	**d**	-1
	e	$\frac{1}{\sqrt{2}}$	**f**	$-\frac{1}{\sqrt{2}}$	**g**	-1	**h**	$\sqrt{2}$
	i	$-\frac{1}{2}$	**j**	$-\frac{\sqrt{3}}{2}$	**k**	$\frac{1}{\sqrt{3}}$	**l**	$\sqrt{3}$
	m	$-\frac{\sqrt{3}}{2}$	**n**	$\frac{1}{2}$				
5	**a**	$\frac{1}{2}$	**b**	$\frac{\sqrt{3}}{2}$	**c**	11	**d**	$\frac{1}{2}$
	e	$-\frac{1}{\sqrt{3}}$	**f**	$-\frac{1}{2}$	**g**	$-\sqrt{2}$		
6	**a**	$-\frac{1}{2}$	**b**	$-\frac{1}{\sqrt{2}}$	**c**	$\sqrt{3}$	**d**	-2
	e	1	**f**	$\frac{1}{2}$	**g**	$-\frac{1}{\sqrt{3}}$	**h**	$-\frac{\sqrt{3}}{2}$
	i	$-\frac{2}{\sqrt{3}}$	**j**	$\frac{1}{\sqrt{3}}$	**k**	$\frac{2}{\sqrt{3}}$		

7	**a**	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	**b**	$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	**c**	$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	**d**	$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
8	**a**	0	**b**	$\frac{\sqrt{3}}{2}$	**c**	$\frac{1}{\sqrt{3}}$	**d**	$\frac{1+\sqrt{3}}{2\sqrt{2}}$
10	**a**	$-\frac{2}{3}$	**b**	$-\frac{2}{3}$	**c**	$-\frac{2}{3}$		
11	**a**	$-\frac{2}{5}$	**b**	$\frac{5}{2}$	**c**	$\frac{2}{5}$		
12	**a**	k	**b**	$-\frac{1}{k}$	**c**	$-k$		
13	**a**	$\frac{\sqrt{5}}{3}$	**b**	$\frac{3}{\sqrt{5}}$	**c**	$-\frac{\sqrt{5}}{3}$		
14	**a**	$-\frac{3}{5}$	**b**	$\frac{3}{4}$	**c**	$\frac{4}{5}$		
15	**a**	$\frac{4}{5}$	**b**	$\frac{3}{4}$	**c**	$-\frac{5}{3}$		
16	**a**	$-k$	**b**	$-\sqrt{1-k^2}$	**c**	$-\frac{k}{\sqrt{1-k^2}}$		
17	**a**	$-\sqrt{1-k^2}$	**b**	$\frac{k}{\sqrt{1-k^2}}$	**c**	$-\frac{1}{\sqrt{1-k^2}}$		
18	**a**	$\sin\theta$	**b**	$\cot\theta$	**c**	1	**d**	1
	e	$\cot\theta$	**f**	$\tan\theta$				
19	**a**	$\frac{\pi}{3}, \frac{2\pi}{3}$	**b**	$\frac{\pi}{3}, \frac{5\pi}{3}$	**c**	$\frac{\pi}{3}, \frac{4\pi}{3}$	**d**	$\frac{5\pi}{6}, \frac{7\pi}{6}$
	e	$\frac{5\pi}{6}, \frac{11\pi}{6}$	**f**	$\frac{7\pi}{6}, \frac{11\pi}{6}$				

Exercise C.8.2

1

a $\sin\alpha\cos\phi + \cos\alpha\sin\phi$ **b** $\cos 3\alpha\cos 2\beta - \sin 3\alpha\sin 2\beta$ **c** $\sin 2x\cos y - \cos 2x\sin y$

d $\cos\phi\cos 2\alpha + \sin\phi\sin 2\alpha$ **e** $\dfrac{\tan 2\theta - \tan\alpha}{1 + \tan 2\theta\tan\alpha}$ **f** $\dfrac{\tan\phi - \tan 3\omega}{1 + \tan\phi\tan 3\omega}$

2

a $\sin(2\alpha - 3\beta)$ **b** $\cos(2\alpha + 5\beta)$ **c** $\sin(x + 2y)$

d $\cos(x - 3y)$ **e** $\tan(2\alpha - \beta)$ **f** $\tan x$

g $\tan\left(\frac{\pi}{4} - \phi\right)$ **h** $\sin\left(\frac{\pi}{4} + \alpha + \beta\right)$ **i** $\sin 2x$

3 **a** $-\frac{56}{65}$ **b** $\frac{33}{65}$ **c** $-\frac{16}{63}$

4 **a** $\frac{16}{65}$ **b** $\frac{63}{65}$ **c** $\frac{56}{33}$

5 **a** $-\frac{5\sqrt{11}}{18}$ **b** $-\frac{7}{18}$ **c** $\frac{5\sqrt{11}}{7}$ **d** $\frac{35\sqrt{11}}{162}$

6 **a** $-\frac{3}{5}$ **b** $-\frac{4}{5}$ **c** $\frac{3}{4}$ **d** $\frac{24}{7}$

7 **a** $\frac{1+\sqrt{3}}{2\sqrt{2}}$ **b** $\frac{1+\sqrt{3}}{2\sqrt{2}}$ **c** $-\frac{1+\sqrt{3}}{2\sqrt{2}}$ **d** $\sqrt{3} - 2$

8 **a** $\frac{2ab}{a^2+b^2}$ **b** $\frac{a^2+b^2}{2ab}$ **c** $\frac{a^4 - 6a^2b^2 + b^4}{(a^2+b^2)^2}$ **d** $\frac{2ab}{b^2-a^2}$

12 $\sqrt{2} - 1$

14 **a** $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$ **b** $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$ **c** $0, \pi, 2\pi, \alpha, \pi \pm \alpha, 2\pi - \alpha, \alpha = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$

15 **a** $R = \sqrt{a^2+b^2}, \tan\alpha = \frac{b}{a}$ **b** 10

16 **a** $R = \sqrt{a^2+b^2}, \tan\alpha = \frac{b}{a}$ **b** −11

18 $2 - \sqrt{3}$

Exercise C.8.3

1	**a**	$\frac{\pi}{4}, \frac{3\pi}{4}$	**b**	$\frac{7\pi}{6}, \frac{11\pi}{6}$	**c**	$\frac{\pi}{3}, \frac{2\pi}{3}$	**d**	$\frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$
	e	$\frac{\pi}{3}, \frac{5\pi}{3}$	**f**	$\frac{5}{4}, \frac{7}{4}, \frac{13}{4}, \frac{15}{4}, \frac{21}{4}, \frac{23}{4}$				
2	**a**	$\frac{\pi}{4}, \frac{7\pi}{4}$	**b**	$\frac{2\pi}{3}, \frac{4\pi}{3}$	**c**	$\frac{\pi}{6}, \frac{11\pi}{6}$	**d**	π
	e	$\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$	**f**	$\frac{3}{2}, \frac{5}{2}, \frac{11}{2}$				
3	**a**	$\frac{\pi}{6}, \frac{7\pi}{6}$	**b**	$\frac{3\pi}{4}, \frac{7\pi}{4}$	**c**	$\frac{\pi}{3}, \frac{4\pi}{3}$	**d**	$4\tan^{-1}2$
	e	$\frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}$	**f**	3				
4	**a**	90°, 330°	**b**	180°,240°	**c**	90°,270°	**d**	65°,335°
	e	$\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$	**f**	$0, \pi, 2\pi$	**g**	$\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$	**h**	$\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$
5	**a**	60°,300°	**b**	$\frac{4\pi}{3}, \frac{5\pi}{3}$	**c**	$\frac{\pi}{6}, \frac{7\pi}{6}$	**d**	23°35',156°25'
	e	$\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$	**f**	$\frac{2\pi}{3}, \frac{5\pi}{3}$	**g**	$\frac{5\pi}{6}, \frac{9\pi}{6}$	**h**	$3.3559^c, 5.2105^c$
	i	$\frac{\pi}{3}, \frac{4\pi}{3}$	**j**	$\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$	**k**	$\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$	**l**	68°12',248°12'
	m	$\frac{\pi}{3}, \frac{5\pi}{3}$	**n**	$\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$	**o**	$\emptyset$		
6	**a**	$-\frac{3\pi}{4}, \frac{\pi}{4}$	**b**	$\pm\frac{\pi}{3}$	**c**	$-\frac{7\pi}{8}, -\frac{3\pi}{8}, \frac{\pi}{8}, \frac{5\pi}{8}$	**d**	$-\frac{\pi}{2}$
	e	$\pm\frac{\pi}{2}$	**f**	$\frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$	**g**	$\frac{\pi}{2}, \frac{3\pi}{2}$	**h**	$\frac{\pi}{2}, \frac{3\pi}{2}$

7 **a** $\frac{\pi}{6},\frac{5\pi}{6},\frac{3\pi}{2},\frac{13\pi}{6},\frac{17\pi}{6},\frac{7\pi}{2}$

b $-2\pi,0,2\pi$

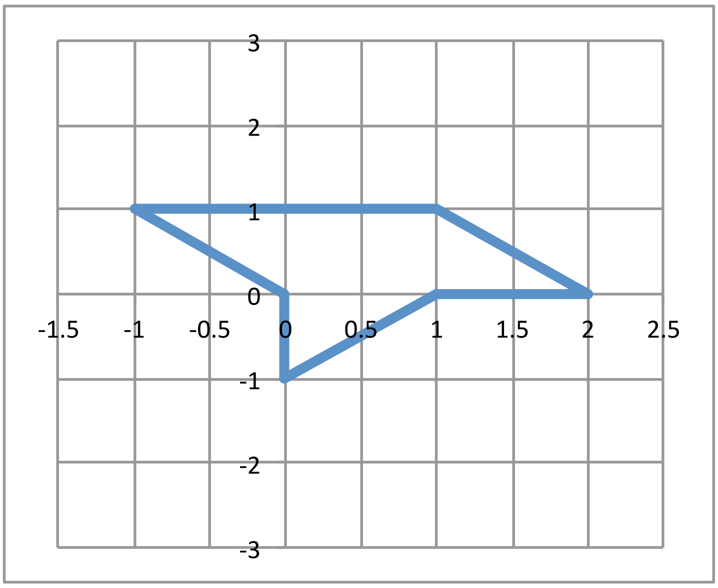
c $-\frac{\pi}{2},\frac{3\pi}{2},\frac{7\pi}{2}$

d $\frac{\pi}{6},\frac{5\pi}{6},\frac{3\pi}{2},\frac{13\pi}{6},\frac{17\pi}{6},\frac{25\pi}{6},\frac{29\pi}{6}$

e $2n\pi\pm\sin^{-1}\left(\frac{1}{3}\right)\pm\frac{\pi}{2},\frac{2(3n\pm1)\pi}{3},n=-1,3$

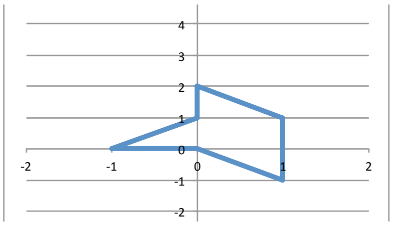
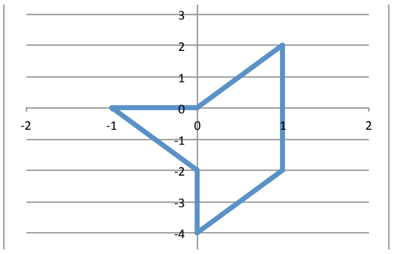
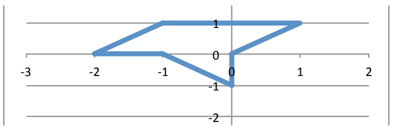
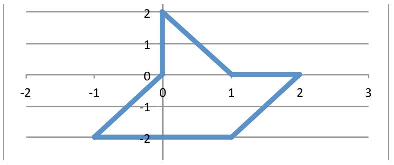
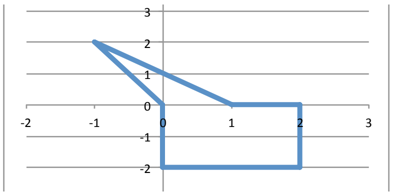
Exercise C.9.1

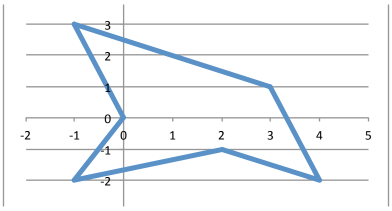
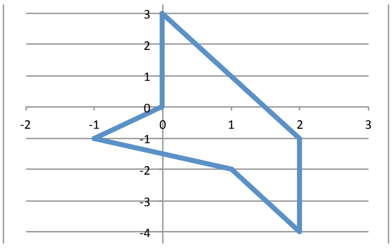
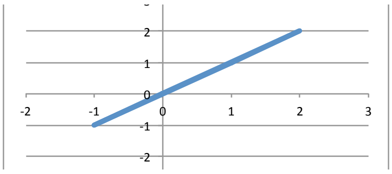
1. a



b 2½ u².

c

Part	Matrix	Transformed Coordinates	Transformed Diagram	Area
i	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 2 & 1 & -1 & 0 & 0 \end{bmatrix}$		2½ u²
ii	$\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & -2 & -4 & -2 & 2 & 0 & 0 \end{bmatrix}$		5 u²
iii	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & -1 \end{bmatrix}$		2½ u²
iv	$\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 2 & 1 & -1 & 0 & 0 \\ 2 & 0 & 0 & -2 & -2 & 0 & 2 \end{bmatrix}$		5 u²
v	$\begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$	$\begin{bmatrix} -1 & 1 & 2 & 2 & 0 & 0 & -1 \\ 2 & 0 & 0 & -2 & -2 & 0 & 2 \end{bmatrix}$		5 u²

Part	Matrix	Transformed Coordinates	Transformed Diagram	Area
vi	$\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$	$\begin{bmatrix} -1 & 2 & 4 & 3 & -1 & 0 & -1 \\ -2 & -1 & -2 & 1 & 3 & 0 & -2 \end{bmatrix}$		12½ u^2
vii	$\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 1 & 2 & 2 & 0 & 0 & -1 \\ -1 & -2 & -4 & -1 & 3 & 0 & -1 \end{bmatrix}$		7½ u^2
viii	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 1 & 2 & 2 & 0 & 0 & -1 \\ -1 & 1 & 2 & 2 & 0 & 0 & -1 \end{bmatrix}$		0 u^2

3. a $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ b $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ c $\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$

d $\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$ e $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ f $\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

g $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ h $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

4. 2 u^2.

5. a $\begin{bmatrix} -3 & 0 \\ 2 & -2 \end{bmatrix}$ b $\begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix}$ c ×6

7. $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

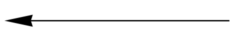
Exercise C.9.2

1. V 2. S 3. S 4. V

5. V 6. V 7. S

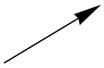
Exercise C.9.3

1 **a**

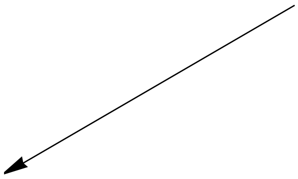


b

c

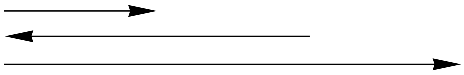


d

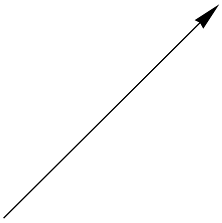


2

a
b
c



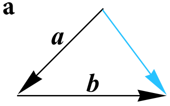
d



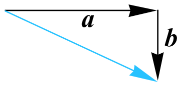
3 **a** {a,b,e,g,u}; {d,f} **b** {d,f}; {a,c}; {b,e} **c** {a,g},{c,g}

d {d,f}, {b,e} **e** {d,f}, {b,e}, {a,c,g}

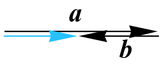
4 a



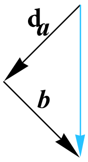
b



c



d



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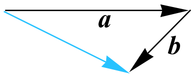
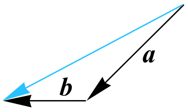
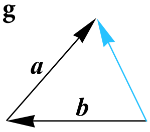


f



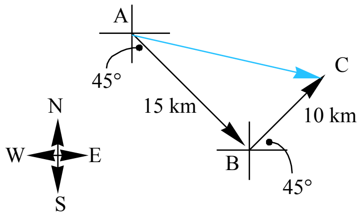
g



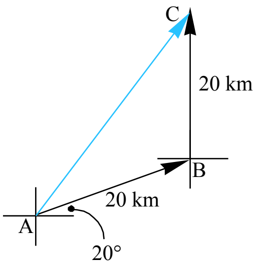
5 **a** AC **b** AB **c** AD **d** BA **e**
0

6 **a** Y **b** N **c** Y **d** Y **e**
N

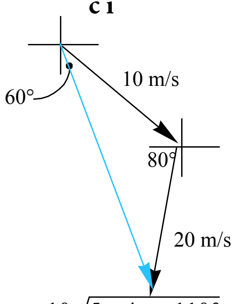
7 a i



b i



c i



ii $\sqrt{325}$ **iv** $20\sqrt{2(1-\cos 110°)}$ **v** $10\sqrt{5-4\cos 110°}$

8 72.11 N, E $33°41'$ N

9 2719 N along river

10 **b i** 200 kph N **ii** 213.6 kph, N $7°37'$ W

11 **b i** 200 **ii** 369.32

Exercise C.9.4

1 **a** $4\boldsymbol{i}+28\boldsymbol{j}-4\boldsymbol{k}$ **b** $12\boldsymbol{i}+21\boldsymbol{j}+15\boldsymbol{k}$ **c** $-2\boldsymbol{i}+7\boldsymbol{j}-7\boldsymbol{k}$ **d** $-6\boldsymbol{i}-12\boldsymbol{k}$

2 **a** $3\boldsymbol{i}-4\boldsymbol{j}+2\boldsymbol{k}$ **b** $-8\boldsymbol{i}+24\boldsymbol{j}+13\boldsymbol{k}$ **c** $18\boldsymbol{i}-32\boldsymbol{j}+\boldsymbol{k}$ **d** $-15\boldsymbol{i}+36\boldsymbol{j}+12\boldsymbol{k}$

3 **a** $\begin{pmatrix}11\\0\\8\end{pmatrix}$ **b** $\begin{pmatrix}-27\\1\\-22\end{pmatrix}$ **c** $\begin{pmatrix}-3\\-6\\12\end{pmatrix}$ **d** $\begin{pmatrix}16\\-1\\14\end{pmatrix}$

4 $\begin{pmatrix}-5\\3\end{pmatrix}$

5 $\begin{pmatrix}-2\\3\end{pmatrix}$, $(-2, 3)$

6 **a** $8\boldsymbol{i}-4\boldsymbol{j}-28\mathbf{k}$ **b** $-19\boldsymbol{i}-7\boldsymbol{j}-16\mathbf{k}$ **c** $-17\boldsymbol{i}+\boldsymbol{j}+22\mathbf{k}$ **d** $40\boldsymbol{i}+4\boldsymbol{j}-20\mathbf{k}$

7 **a** $\begin{pmatrix}20\\1\\25\end{pmatrix}$ **b** $\begin{pmatrix}12\\2\\16\end{pmatrix}$ **c** $\begin{pmatrix}-4\\-38\\-32\end{pmatrix}$ **d** $\begin{pmatrix}-20\\-22\\-40\end{pmatrix}$

8 $A = -4$, $B = -7$

9 **a** $(2, -5)$ **b** $(-4, 3)$ **c** $(-6, -5)$

10 Depends on basis used. Here we used: East as $\boldsymbol{i}$, North $\boldsymbol{j}$ and vertically up $\boldsymbol{k}$

b $D = 600\boldsymbol{i}-800\boldsymbol{j}+60\boldsymbol{k}$, $A = -1200\mathbf{i}-300\boldsymbol{j}+60\boldsymbol{k}$ **c** $1800\boldsymbol{i}-500\boldsymbol{j}$

Exercise C.9.5

1 **a** 4 **b** −11.49 **c** 25

2 **a** 12 **b** 27 **c** −8 **d** −49

f 4 **g** −21 **h** 6 **i** −4

3 **a** 79° **b** 108° **c** 55° **d** 50°

e 74° **f** 172° **g** 80° **h** 58°

4 **a** −8 **b** 0.5

5 **a** −6 **b** 2 **c** Not possible **d** 5

e Not possible **f** 0

6 **a** $4-2\sqrt{3}$ **b** $2\sqrt{3}-4$ **c** $14-2\sqrt{3}$ **d** Not possible

7 1

8 105.2°

9 $x = -\frac{16}{7}, y = -\frac{44}{7}$

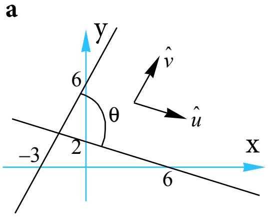
10 $\pm\frac{1}{\sqrt{11}}(-\boldsymbol{i} + \mathbf{j} + 3\boldsymbol{k})$

12 **a** $\lambda(-16\boldsymbol{i} - 10\boldsymbol{j} + \boldsymbol{k})$ **b** e.g. $\boldsymbol{i} + \boldsymbol{j} + \frac{3}{7}\boldsymbol{k}$

14 $\boldsymbol{a} \perp \boldsymbol{b} - \boldsymbol{c}$ if $\boldsymbol{b} \neq \boldsymbol{c}$ or $\boldsymbol{b} = \boldsymbol{c}$

16 **a** $\frac{1}{3}$ **b** $\frac{1}{\sqrt{3}}$

17 **a**



b i $\hat{\boldsymbol{u}} = \frac{1}{\sqrt{10}}(3\boldsymbol{i} - \mathbf{j})$ **ii** $\hat{\boldsymbol{v}} = \frac{1}{\sqrt{5}}(\boldsymbol{i} + 2\mathbf{j})$

c 81.87°

18 $\frac{1}{2}(-\boldsymbol{i} + \mathbf{j} + \sqrt{2}\boldsymbol{k})$

23 **a** Use $\boldsymbol{i}$ as a 1 km eastward vector and $\boldsymbol{j}$ as a 1 km northward vector.

b $\overrightarrow{\mathrm{WD}} = 4\boldsymbol{i} + 8\boldsymbol{j}$, $\overrightarrow{\mathrm{WS}} = 13\boldsymbol{i} + \boldsymbol{j}$ and $\overrightarrow{\mathrm{DS}} = 9\boldsymbol{i} - 7\boldsymbol{j}$ **c** $\frac{1}{\sqrt{80}}(4\boldsymbol{i} + 8\boldsymbol{j})$

d $\frac{\mathrm{d}}{\sqrt{80}}(4\boldsymbol{i} + 8\boldsymbol{j})$ **e** $3\boldsymbol{i} + 6\boldsymbol{j}$

Exercise C.9.6

1 **a i** $r = \boldsymbol{i} + 2\boldsymbol{j}$ **ii** $r = -5\boldsymbol{i} + 11\boldsymbol{j}$ **iii** $r = 5\boldsymbol{i} - 4\boldsymbol{j}$ **b** line joins (1, 2) and (5, –4)

2 **a** $r = 2\boldsymbol{i} + 5\boldsymbol{j} + \lambda(3\boldsymbol{i} - 4\boldsymbol{j})$ **b** $r = -3\boldsymbol{i} + 4\boldsymbol{j} + \lambda(-\boldsymbol{i} + 5\boldsymbol{j})$ **c** $r = \boldsymbol{j} + \lambda(7\boldsymbol{i} + 8\boldsymbol{j})$

d $r = \boldsymbol{i} - 6\boldsymbol{j} + \lambda(2\boldsymbol{i} + 3\boldsymbol{j})$ **e** $r = \begin{pmatrix}-1\\-1\end{pmatrix} + \lambda\begin{pmatrix}-2\\10\end{pmatrix}$ or $r = -\boldsymbol{i} - \boldsymbol{j} + \lambda(-2\boldsymbol{i} + 10\boldsymbol{j})$

f $r = \begin{pmatrix}1\\2\end{pmatrix} + \lambda\begin{pmatrix}5\\1\end{pmatrix}$ or $r = \boldsymbol{i} + 2\boldsymbol{j} + \lambda(5\boldsymbol{i} + \boldsymbol{j})$

3 **a** $r = 2\boldsymbol{i} + 3\boldsymbol{j} + \lambda(2\boldsymbol{i} + 5\boldsymbol{j})$ **b** $r = \boldsymbol{i} + 5\boldsymbol{j} + \lambda(-3\boldsymbol{i} - 4\boldsymbol{j})$ **c** $r = 4\boldsymbol{i} - 3\boldsymbol{j} + \lambda(-5\boldsymbol{i} + \boldsymbol{j})$

4 **a** $r = 9\boldsymbol{i} + 5\boldsymbol{j} + \lambda(\boldsymbol{i} - 3\boldsymbol{j})$ **b** $r = 6\boldsymbol{i} - 6\boldsymbol{j} + \mathrm{t}(-4\boldsymbol{i} - 2\boldsymbol{j})$

c $r = -\boldsymbol{i} + 3\boldsymbol{j} + \lambda(-4\boldsymbol{i} + 8\boldsymbol{j})$ **d** $r = \boldsymbol{i} + 2\boldsymbol{j} + \mu\left(\frac{1}{2}\boldsymbol{i} - \frac{1}{3}\boldsymbol{j}\right)$

5 **a** $x = -8 + 2\mu$, $y = 10 + \mu$ **b** $x = 7 - 3\mu$, $y = 4 - 2\mu$

c $x = 5 + 2.5\mu$, $y = 3 + 0.5\mu$ **d** $x = 0.5 - 0.1t$, $y = 0.4 + 0.2t$

6 **a** $\frac{x-1}{3} = y - 3$ **b** $\frac{x-2}{-7} = \frac{y-4}{-5}$

c $x + 2 = \frac{y+4}{8}$ **d** $x - 0.5 = \frac{y-0.2}{-11}$

e $x = 7$

7 **a** $r = 2\mathbf{j} + t(3\mathbf{i} + \mathrm{j})$ **b** $r = 5\mathbf{i} + t(\mathbf{i} + \mathbf{j})$ **c** $r = -6\mathbf{i} + t(2\mathbf{i} + \mathbf{j})$

8 **a** $6\mathbf{i} + 13\mathbf{j}$ **b** $-\frac{16}{3}\mathbf{i} - \frac{28}{3}\mathbf{j}$

9 $r = 2\mathbf{i} + 7\mathbf{j} + t(4\mathbf{i} + 3\mathbf{j})$

11 **a** (4, −2), (−1, 1), (9, −5) **b** −2 **d** $r = 4\mathbf{i} - 2\mathbf{j} + \lambda(-5\mathbf{i} + 3\mathbf{j})$ **e i** $\mathrm{M} \parallel \mathrm{L}$ **ii** $\mathrm{M} = \mathrm{L}$

12 $4x + 3y = 11$

13 **a** $\frac{-3}{\sqrt{13}}, \frac{2}{\sqrt{13}}$ **b** $\frac{4}{5}, \frac{3}{5}$

14 **b** ii and iii

15 (−83, −215)

16 $r = \frac{k}{7}(19\mathbf{i} + 20\mathbf{j})$

17 **a** $\left(\frac{92}{11}, \frac{31}{11}\right)$ **b** Ø **c** Lines are coincident, all points are common.

Exercise C.9.7

1 **a** $r = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$ **b** $r = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k} + t(-2\mathbf{i} + \mathbf{k})$

2 **a** $r = 2\mathbf{i} + 5\mathbf{k} + t(\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$ **b** $r = 3\mathbf{i} - 4\mathbf{j} + 7\mathbf{k} + t(4\mathbf{i} + 9\mathbf{j} - 5\mathbf{k})$

c $r = 4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k} + t(7\mathbf{i} + 7\mathbf{k})$

3 **a** $\frac{x}{3} = \frac{y-2}{4} = \frac{z-3}{5}$ **b** $\frac{x+2}{5} = \frac{z+1}{-2}, y = 3$ **c** $x = y = z$

4 $\begin{aligned} x &= 5-7t \\ y &= 2+2t \\ z &= 6-4t \end{aligned}$ $\boldsymbol{r} = \begin{pmatrix} 5 \\ 2 \\ 6 \end{pmatrix} + t\begin{pmatrix} -7 \\ 2 \\ -4 \end{pmatrix}$ $\frac{x-5}{-7} = \frac{y-2}{2} = \frac{z-6}{-4}$

5 $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$

6 **a** $\begin{aligned} x &= 2+3t \\ y &= 5+t \\ z &= 4+0.5t \end{aligned}$ **b** $\begin{aligned} x &= 1+1.5t \\ y &= t \\ z &= 4-2t \end{aligned}$

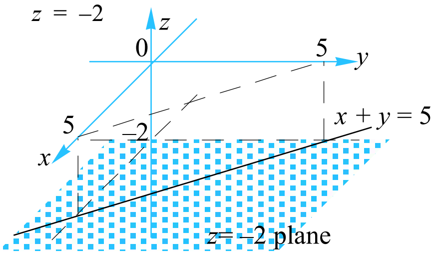
c $\begin{aligned} x &= 3-t \\ y &= 2-3t \\ z &= 4+2t \end{aligned}$ **d** $\begin{aligned} x &= 1+2t \\ y &= 3+2t \\ z &= 2+0.5t \end{aligned}$

7 **a** $\frac{x-4}{3} = \frac{y-1}{-4} = \frac{z+2}{-2}$ **b** $x = 2, y = \frac{z-1}{-3}$

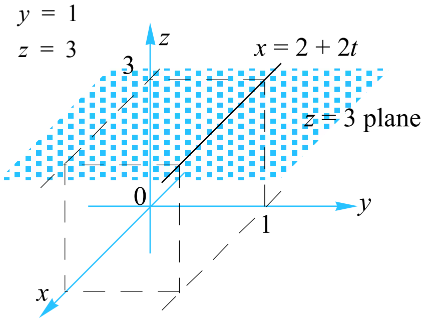
9 **a** $\frac{x+1}{2} = y-3 = \frac{z-5}{-1}$ **b** $\frac{x-2}{2} = \frac{z-1}{-2}, y = 1$

10 **a** $(1, -1, 0)$ **b** $a = 15, b = -11$

11 a $\begin{aligned} x &= 1+t \\ y &= 4-\text{t} \\ z &= -2 \end{aligned}$



b $\begin{aligned} x &= 2+2t \\ y &= 1 \\ z &= 3 \end{aligned}$



12 $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0.5 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1.5 \\ 1 \end{pmatrix}$. Line passes through (1, 0.5, 2) and is parallel to the vector $2\boldsymbol{i} - \frac{3}{2}\boldsymbol{j} + \boldsymbol{k}$

13 **a** 54.74° **b** 82.25° **c** 57.69°

14 **a** (4, 10.5, 15) **b** Does not intersect.

15 **a** $L: x = \frac{y-2}{2} = \frac{z}{5}, M: \frac{x+1}{2} = \frac{y+1}{3} = \frac{z-1}{-2}$ **b** $\emptyset$ **c** 84.92°

d i (0, 2, 0) **ii** $\left(0, \frac{1}{2}, 0\right)$

18 $\frac{x}{4} = \frac{y}{9} = \frac{z}{3}$

19 $k = -\frac{7}{2}$

20 64°

21 3 or −2

22 $12\boldsymbol{i} + 6\boldsymbol{j} - 7\boldsymbol{k}$ (or any multiple thereof)

23 Not parallel. Do not intersect. Lines are skew.

Exercise C.9.8

1 t=2

2 (1,1,1), no

3 ~21

4 a $P_A(t)=\begin{pmatrix} 0 \\ 100 \end{pmatrix}+t\begin{pmatrix} \sqrt{5} \\ -\sqrt{5} \end{pmatrix}$, $P_B(t)=\begin{pmatrix} 0 \\ 0 \end{pmatrix}+t\begin{pmatrix} 10 \\ 0 \end{pmatrix}$ b 98m

5. a $P_{train}(t)=\begin{pmatrix} 0 \\ 7 \end{pmatrix}+t\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ $P_{car}(t)=\begin{pmatrix} 5.25 \\ 7 \end{pmatrix}+t\begin{pmatrix} 0.25 \\ -1 \end{pmatrix}$

b 1.75 units c they collide at t = 3 (unless the driver looks where sh/e is going!)

6. a $P_A(t)=\begin{pmatrix} 0 \\ 1 \end{pmatrix}+t\begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $P_B(t)=\begin{pmatrix} 1 \\ 8 \end{pmatrix}+t\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

b 3.16 u

c No

7. a east (positive) in metres, north (positive) in metres, altitude in hundreds of feet.

This mixture of units is quite common. Aviation measures distances in nautical miles and altitude in feet.

b $P_{YMAD}(t)=\begin{pmatrix} 0 \\ 4 \\ 7 \end{pmatrix}+t\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ $P_{YMBC}(t)=\begin{pmatrix} 2 \\ 8 \\ 5 \end{pmatrix}+t\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

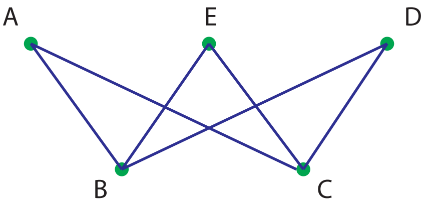
c $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

d Collision at t = 2 unless pilots see one another - or air traffic control or TCAS warns them.

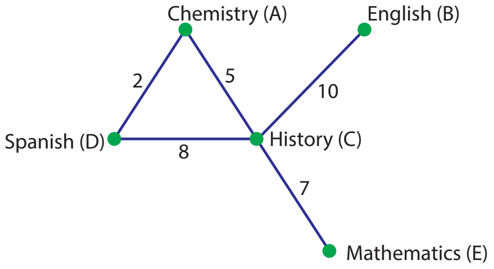
Exercise C10.1

1. A sample graph with 5 vertices can only have at the most 10 edges. So a graph with 5 vertices and 12 edges cannot be a simple graph.

2.



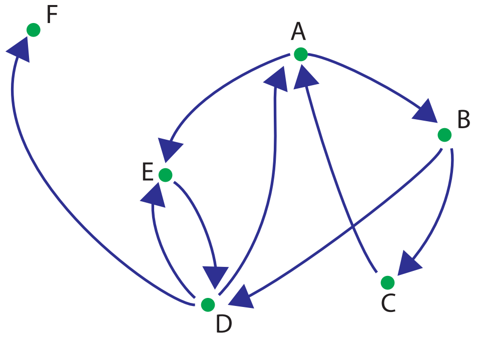
3. a



b

$$\begin{array}{c|ccccc} & A & B & C & D & E \\ A & 0 & 0 & 5 & 2 & 0 \\ B & 0 & 0 & 10 & 0 & 0 \\ C & 5 & 10 & 0 & 8 & 7 \\ D & 2 & 0 & 8 & 0 & 0 \\ E & 0 & 0 & 7 & 0 & 0 \end{array}$$

4. a



b Yes, it is possible if a traveller takes the following route B, C, A, E, D, F.

c AE and ABDE.

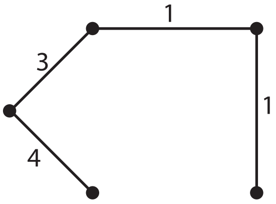
d

$$\begin{array}{c|cccccc} & A & B & C & D & E & F \\ A & 0 & 1 & 0 & 0 & 1 & 0 \\ B & 0 & 0 & 1 & 1 & 0 & 0 \\ C & 1 & 0 & 0 & 0 & 0 & 0 \\ D & 1 & 0 & 0 & 0 & 1 & 1 \\ E & 0 & 0 & 0 & 1 & 0 & 0 \\ F & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

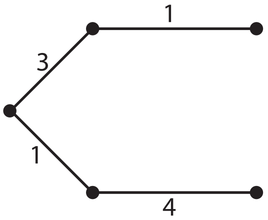
e There are two ways.

Exercise C10.2

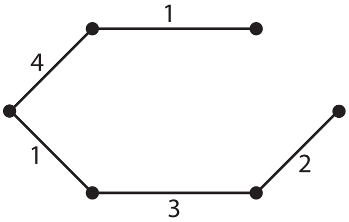
1. a



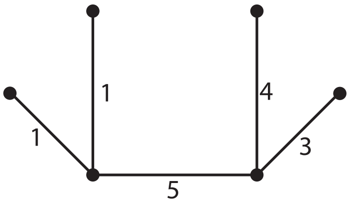
b



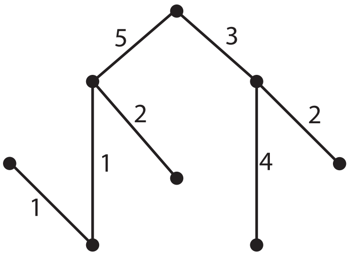
c



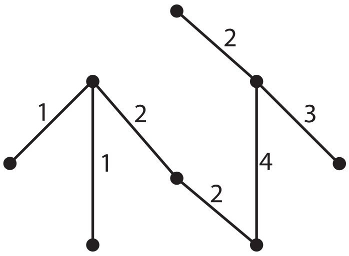
d



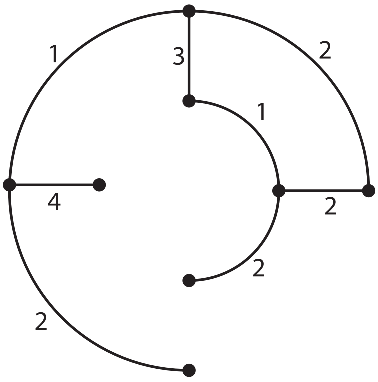
e



f



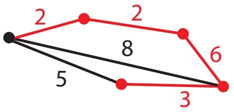
g



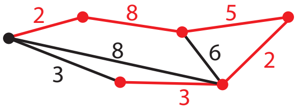
Exercise C.10.3

1.

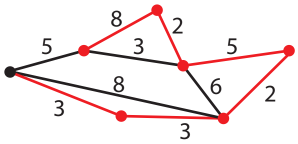
a



b



c



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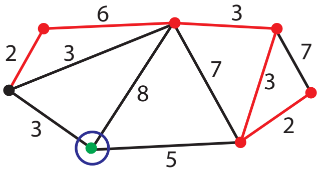

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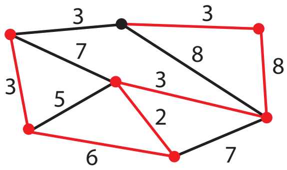

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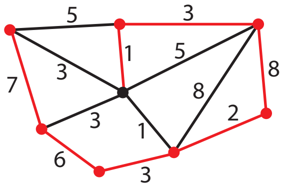
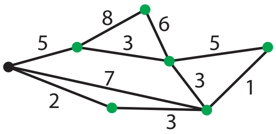


Lower Bound

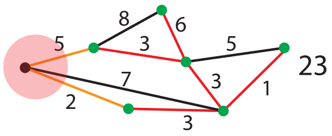
Upper Bound
(not unique)

3.

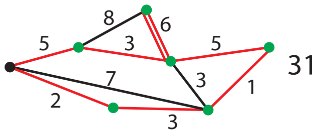
a



a



a



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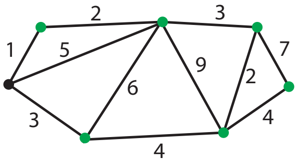

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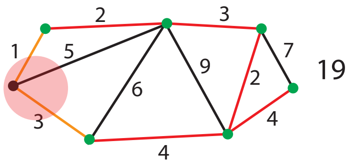

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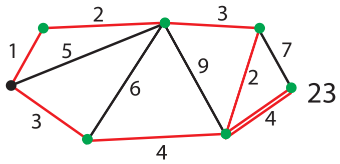
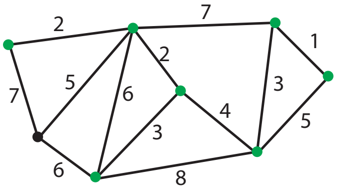


c



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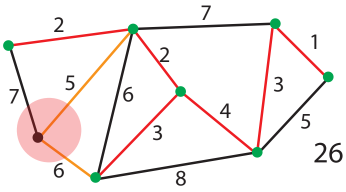

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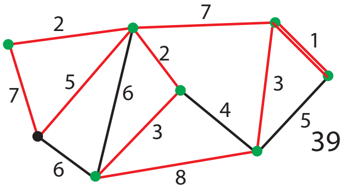

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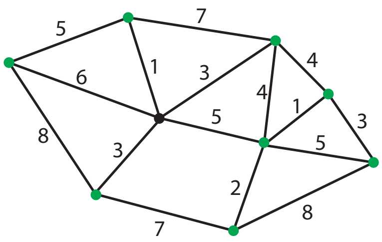

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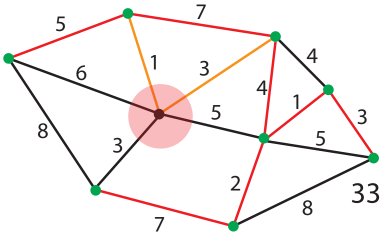

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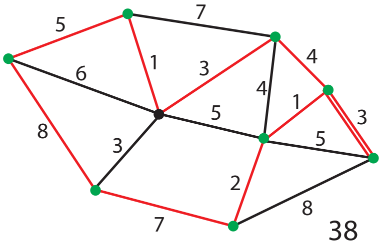
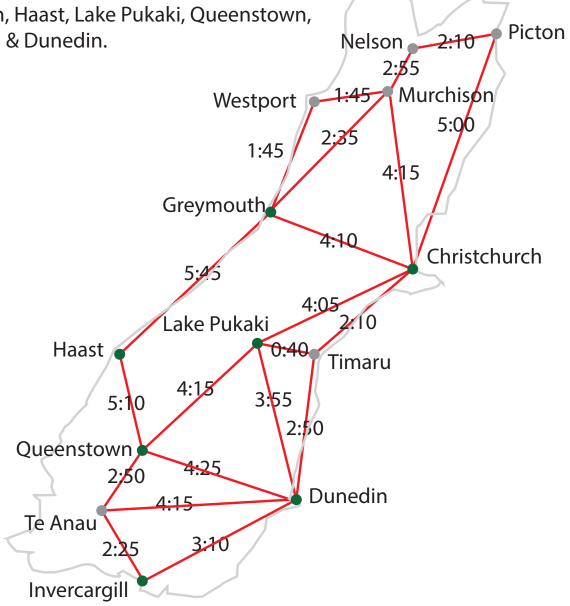
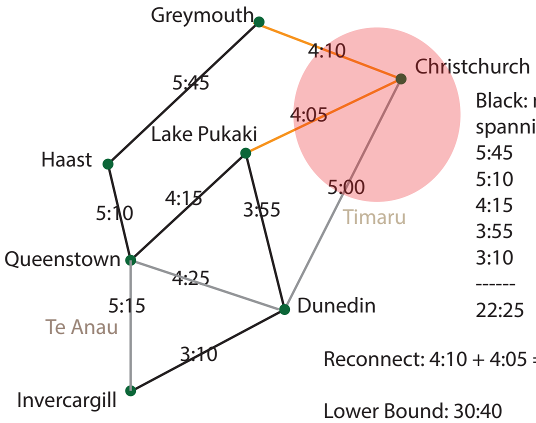


The Ortel Family will fly into and out of Christchurch where they will hire a car. They intend to visit Greymouth, Haast, Lake Pukaki, Queenstown, Invercargill & Dunedin.



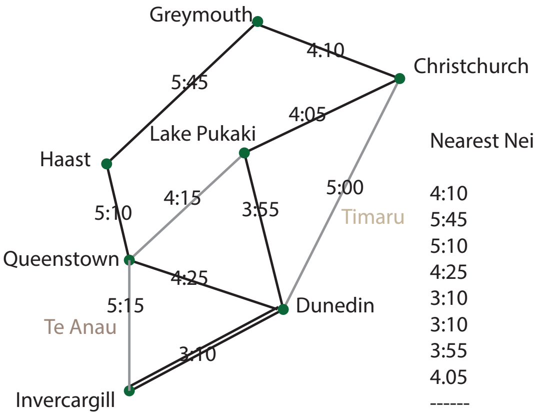


Black: minimum spanning tree
5:45
5:10
4:15
3:55
3:10

22:25

Reconnect: 4:10 + 4:05 = 8:15

Lower Bound: 30:40



Nearest Neighbour

4:10
5:45
5:10
4:25
3:10
3:10
3:55
4.05

35:50 (upper bound)

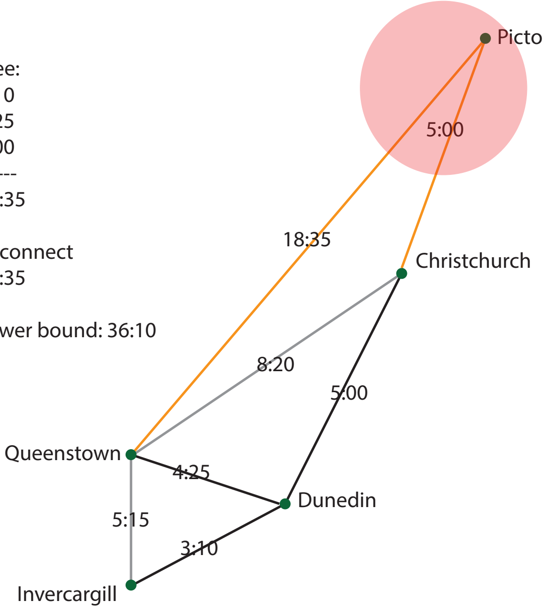
The Zorn Family will arrive at Picton on the ferry from Wellington. There they will rent a Campervan. They are particularly interested in visiting Christchurch, Dunedin, Invercargill and Queenstown. At the end or the trip they must return their van to Picton.

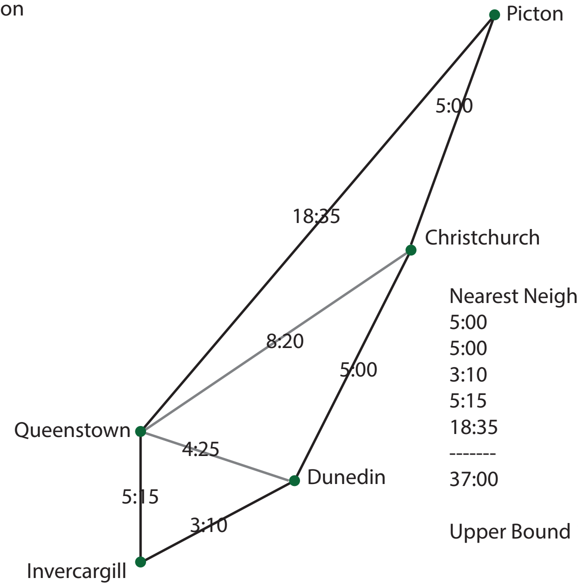
Tree:
3:10
4:25
5:00

12:35

Reconnect
23:35

Lower bound: 36:10





Nearest Neighbour
5:00
5:00
3:10
5:15
18:35

37:00

Upper Bound 13:00

The Vogelgesang family have no chance of implementing their plan. This, however, remains a challenging question if you follow up to ask what is the biggest list of towns they can manage?

Exercise D.9.1

1. a High reliability and validity. b Unreliable but valid
   c Reliable, but invalid d Unreliable and invalid.

2. a 1 b 5 c 0 d 11 e 11.

3. a 1 b 12.96 c the critical value is 3.841 < 12.96 so the 'fair' hypothesis is rejected.

4 a 5 b 3.8 c the critical value is 11.07 > 3.8 so the 'fair' hypothesis is accepted.

5. a

Score	Frequency	Probability	Exp	Obs-Exp	(Obs-Exp)2	/exp
2	9	0.028	2.778	6.222	38.716	13.938
3	12	0.056	5.556	6.444	41.531	7.476
4	13	0.083	8.333	4.667	21.778	2.613
5	12	0.111	11.111	0.889	0.790	0.071
6	17	0.139	13.889	3.111	9.679	0.697
7	16	0.167	16.667	-0.667	0.444	0.027
8	12	0.139	13.889	-1.889	3.568	0.257
9	13	0.111	11.111	1.889	3.568	0.321
10	9	0.083	8.333	0.667	0.444	0.053
11	3	0.056	5.556	-2.556	6.531	1.176
12	2	0.028	2.778	-0.778	0.605	0.218

b 11 c 26.85 c the critical value is 19.67 < 26.85 so the 'fair' hypothesis is rejected.

6. a

Height Range	Probability	Exp
[160,165)	0.0605	6.050
[165,170)	0.1609	16.090
[170,175)	0.2625	26.250
[175,180)	0.2625	26.250
[180,185)	0.1609	16.090
[190,195)	0.0605	6.050
[195,200)	0.0139	1.390

b 6

c $\chi^2 = 1.07$

Height Range	Frequency	Probability	Exp	Obs-Exp	$(\text{Obs-Exp})^2$	$(\text{obs-Exp})^2/\text{exp}$
[160,165)	7	0.0605	6.050	0.950	0.903	0.149
[165,170)	18	0.1609	16.090	1.910	3.648	0.227
[170,175)	27	0.2625	26.250	0.750	0.563	0.021
[175,180)	23	0.2625	26.250	-3.250	10.563	0.402
[180,185)	16	0.1609	16.090	-0.090	0.008	0.001
[190,195)	7	0.0605	6.050	0.950	0.903	0.149
[195,200)	2	0.0139	1.390	0.610	0.372	0.268

d the critical value is 12.6 > 1.07 so the ‘normal distribution’ hypothesis is accepted.

7. The mean weight is 499 gms which seems fine. However there is evidently a big spread. The standard deviation is 27.7 gm which is big. The inspectors might like to perform a study comparing the data with a normal distribution, but that is scarcely necessary on the strength of the standard deviation alone.

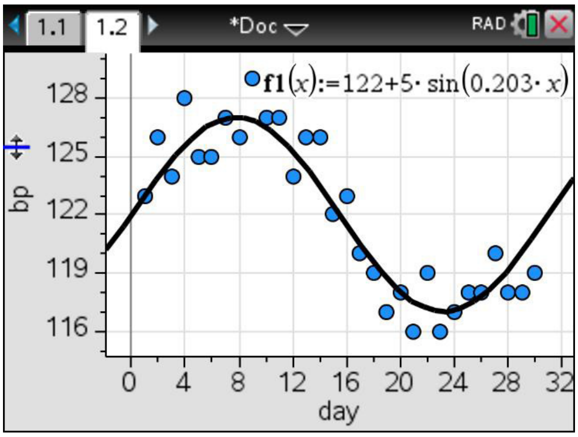
8. Both sets are blatantly leading and utterly hopeless for finding the truth!

Exercise D.10.1

1. a 0.864 b ~1 c 0.77
d 0.95 e 0.84 f 0.58
g 0.45

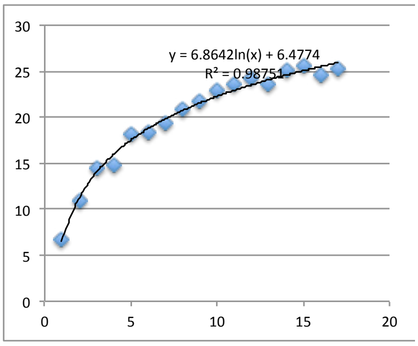
2. The data is, visually, sinusoidal. However, we have not succeeded in persuading technology to produce a good sinusoidal model

The model shown fits the data well and supports the biorhythm idea.



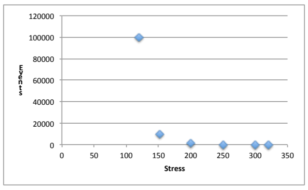
$f1(x):=122+5\cdot\sin(0.203\cdot x)$

3.

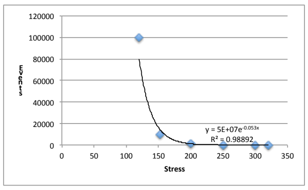


4. a The presentation as powers of 10 suggests an exponential/logarithmic model.

b



c The exponential version of the model is Events = $5\times10^7 e^{-0.05\times\text{stress}}$. The relation becomes logarithmic if the variables are transposed.



d $R^2 = 0.99$ indicating a very good fit to the data.

e 124 000 cycles

f 78 MPa.

5. No, the correlation is too weak for this extrapolation.

6. Using x = Price and y = profit ($,000)

a $y = -x^2 + 12x - 32$

b $y=11.518\log_e(x) - 15.664$

c R^2 is 1 for the quadratic model and 0.985 for the logaritmic model. Both are good fits with the quadratic model being slightly the better.

d The prediction is for a maximum profit of $4,000 at a price of $6.

e This predicts slowly increasing profits as price is increased - but with diminishing returns. Commonsense suggests that the quadratic model is more realistic.

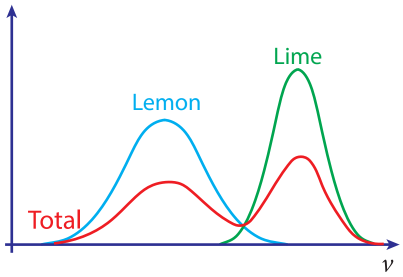
f,g Answers will vary.

h Such goods are known as 'Giffen Goods'. Luxury cars are an example. Part of their attraction is their high price. For a more subtle reason, bread and rice are also Giffen goods. Why?

Exercise D.11.1

1. Mean = 116.76 STD = 12.15 kJ

2.



3. $\mu = 4.2$, $\sigma = 0.6$

4. $\mu = 104$ kph, $\sigma = 4.8$ kph

5. $\mu = 61.8$ g, $\sigma = 1.58$ g

6. 68 kph ≈ 18.89 mps 0.05294 s b 17 m/s c $\mu =$ 61.2 kph $\sigma = 4.54$ kph

Exercise D.11.2

1. a) $\mu \approx 340.2$ g, $\sigma \approx 16$ g

   b) $\mu \approx 340.2$ g. We cannot infer σ, as the population is not normally distributed, and we only have one sample (so we can't use the Central Limit Theorem).

2, a $\mu \approx 7.61$ Kg, $\sigma \approx 0.96$ Kg

   b The average weight is down by nearly 200g. This is unlikely to have happened by chance - Further investigation or intervention may be a good idea

3. a) No.

   The central limit theorem only applies to sets of samples. Student A has only taken one sample. The CLT works best for sample sizes of 30 or more. Student A has only recorded 15 values.

   b S_{n-1} is the best available indicator for σ. $\sigma \approx 1.9$ o/L

4. $\sigma = 14.51810542$. The swimming pools have a higher contamination standard deviation than what is acceptable, so her conclusion is invalid (or at the very least, warrants additional investigation)

5. Yes - $\sigma_{\bar{x}} = \sigma$. The secret codeword is 0.14 g Kg^{-1}.

Exercise D.12.1

1 a $P(X=x)=\dfrac{e^{-2}.2^{x}}{x!}, x=0,1,2,\ldots$

b i 0.1353 ii 0.2707 iii 0.5940 iv 0.4557

2 a 0.0383 b 0.1954

3 a 0.2052 b 0.9179

4 a 0.2623 b 0.8454

5 a 0.0265 b 0.0007

6 a 0.1889 b 0.7127

7 a 0.7981 b 0.2019 c 0.1835

8 a 0.2661 b 0.5221

9 0.1912

10 a 0.3504 b 0.6817

11 a 0.00127 b 0.0500

12 a 0.1804 b 0.0166 c 0.3233

13 a 0.8131; 0.5511 No

14 14. 0.4781

15 a 0.3679 b 0.2642 c 0.2135

16 a i p ii $-p \ln p$ iii $-p+p\ln p$ c 0.4785

Exercise D.13.1

1. H_0: $\mu = 250$, H1: $\mu > 250$, df = 16, $t_c = 1.34$, t = 1.37 > t_c, reject H_0.
2. H_0: $\mu = 2.2$, H1: $\mu > 2.2$, df = 10, $t = 1.93 > t_c = 1.81$, reject H_0.
3. H_0: $\mu = 57$, H1: $\mu \neq 57$, df = 7, $t = 1.51$, $t_c = \pm 1.89$, $-1.89 < t < 1.89$, accept H_0.
4. H_0: $\mu = 20$, H1: $\mu \neq 20$, df = 16, $t = -1.14$, $t_c = \pm 1.74$, $-1.74 < t < 1.74$, accept H_0.
5. H_0: $\mu = 57$, H1: $\mu \neq 57$, df = 7, $t = 1.51$, $t_c = \pm 1.89$, $-1.89 < t < 1.89$, accept H_0.

Exercise D.13.2

1. $p = 0.428\%$ is below the 1% threshold, so accept H_1, i.e. the scores have improved.
2. $p = 0.000144\ \%$ is way below the 1% and 10% thresholds, so accept H_1 , i.e. the murder rate has increased at the 1% & 10% significance levels.
3. $p = 54.2\ \%$ is above the 5% threshold, so accept H_0 , i.e. the time taken with each model is the same.

Exercise D.13.3

1. a The p-value ≈ 38.3%, which is above the threshold of 10%, so we accept H_0 that $p = 0.3$,

   b The p-value ≈ 49.7%, which is above the threshold of 10%, so we accept H_0 that $p = 0.8$.

   c The p-value ≈ 54.8%, which is above the threshold of 10%, so we accept H_0 that $p = 0.4$.

   d The p-value ≈ 44.2%, which is above the threshold of 10%, so we accept H_0 that $p = 0.2$.

2.
   a The p-value = 3.55%, which is below the threshold of 5%, so we accept H_1 that less than 3 out of every 10 callers to the computer help desk must wait more than half an hour before receiving attention.

   b The p-value = 3.55%, which is above the threshold of 2%, so we accept H_0 that it is still the case that 3 out of every 10 callers to the computer help desk must wait more than half an hour before receiving attention.

3. The p-value = 7.30%, which is above the threshold of 5%, so we accept H0 that the student got 9 correct answers by guessing all the answers.

Exercise D.13.4

1. $H_0 = 3$, $H_1 < 3$, $P(X \leq 8) = 0.15$, so reject H_1 that the machine is less likely to break down.
2. $H_0 = 5$, $H_1 > 5$, $P(X \geq 65) = 0.023$, so accept H_1 that the spawning run has increased.

Exercise D.13.5

1. a A Type I error is rejecting a true H_0.

   b A Type II error is accepting a false H_0.

2. There are many correct answers.

   a $\mu = 0.51$, $\bar{x} = 0.49$, H_0 is correctly accepted

   b $\mu = 0.53$, $\bar{x} = 0.54$, H_0 is correctly rejected

   c $\mu = 0.51$, $\bar{x} = 0.53$, H_0 is incorrectly rejected

   d $\mu = 0.53$, $\bar{x} = 0.51$, H_0 is incorrectly accepted

3. The probability of a Type I error is the significance level = 0.1

Exercise D.13.6

1. We could increase the sample size. This has the advantage of reducing β without increasing α and the disadvantage of costing more. Or we could increase the significance level. This has the disadvantage of increasing α and the advantage of not costing more.

2. a 0.808, b 0.0703

3. a 0.513, b 0.112

Exercise D.13.7

1. $\alpha = 0.1$, $\beta = 0.688$

2. a SL = 19.4%, b $\alpha = 0.194$, c $\beta = 0.417$

Exercise D.13.8

1. Power is the probability of correctly rejecting H0 = 1 - β

2. 0 to 1

3. High power is better.

4. 1 – 0.32 = 0.68

Exercise D.14.1

1 a [0.4305,0.5695] b [0.1944,0.8056] c [0.4359,0.5641]

d [0.576,0.424] e [0.3592,0.229,0.4118] f [0.3496,0.3154,0.435]

g [0.1817,0.4347,0.3836] h [0.1887,0.5373,0.274]

2. a [0.25,0.75] b $\frac{2}{3} \quad \frac{1}{3}$ c $\begin{bmatrix} \frac{2}{11} & \frac{43}{99} & \frac{38}{99} \end{bmatrix}$

d $\begin{bmatrix} \frac{7}{18} & \frac{1}{6} & \frac{4}{9} \end{bmatrix}$ e [0.08537,0.208841,0.38872,0.31707]]

3. a $\begin{bmatrix} 0.5 & 0.5 \end{bmatrix}, \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$

b [0.6095,0.3905].

c Two thirds TV1, one third TV2.

4. a $\begin{bmatrix} 0.6 & 0.4 & 0 \end{bmatrix}, \begin{bmatrix} 0.8 & 0 & 0.2 \\ 0 & 0.9 & 0.1 \\ 0 & 0 & 1 \end{bmatrix}$

b [0.48,0.36,0.16]

c Everyone will switch to waterobics.

5. The values for vertices A & C will exchange at each iteration. Vertex B retains its initial value,

6. a $\begin{bmatrix} 0.3 & 0.5 & 0.2 \end{bmatrix}, \begin{bmatrix} 0.8 & 0.05 & 0.15 \\ 0 & 0.9 & 0.1 \\ 0.05 & 0.15 & 0.8 \end{bmatrix}$

b [0.18475,0.5009125,0.3143375]

c 8% butter, 57% margarine, 35% polyunsat.

7. Note that this is a 6 week campaign. The long term is not asked for: Populus 30%, Monetus 10%, Civius 38% and Ecomus 23%.

8. In this case, the population movements are very slow and the migration patterns are unlikely to stay the same long enough to get near equilibrium. After 5 years, the approximate populations are: Altair (pop. 25 000), Beta (pop. 34 000), Canis (pop. 11 000) and Draco (pop. 29 000)

Exercise E.12.1

1.	a	$x^4-\frac{x^2}{2}+x+c$	b	$\frac{2x^{5/2}}{5}-2x+c$	c	$\frac{x^3}{3}+2x-\frac{1}{x}+c$
	d	$\frac{2x\sqrt{x^5}}{7}+c$	e	$6\sqrt{x}+c$	f	$2x^3-\frac{x^2}{2}-x+c$
2.	a	$12\sqrt{x}+c$	b	$6x+2\sqrt{x}+c$	c	$x-\frac{2}{x}-\frac{1}{3x^3}+c$
	d	$\frac{3x\sqrt[3]{x}}{4}+c$	e	$\frac{3x^{5/3}}{5}-\frac{3x^{4/3}}{2}+x+c$	f	$\frac{-7}{5x^5}+c$
3.	a	$\frac{3x^{5/3}}{5}+c$	b	$\frac{3x^{7/3}}{7}+c$	c	$\frac{2x\left(6x^{2/3}+15x^{1/3}+10\right)}{5}+c$
	d	$\frac{(x+1)^5}{5}+c$	e	$\frac{(2x-1)^5}{10}+c$	f	$\frac{5x^{6/5}}{6}-\frac{3x^{4/3}}{4}+c$

Exercise E.12.2

1	**a**	$\frac{1}{5}e^{5x}+c$	**b**	$\frac{1}{3}e^{3x}+c$	**c**	$\frac{1}{2}e^{2x}+c$		
	d	$10e^{0.1x}+c$	**e**	$-\frac{1}{4}e^{-4x}+c$	**f**	$-e^{-4x}+c$		
	g	$-0.2e^{-0.5x}+c$	**h**	$-2e^{1-x}+c$	**i**	$5e^{x+1}+c$		
	j	$e^{2-2x}+c$	**k**	$3e^{x/3}+\mathrm{c}$	**l**	$2\sqrt{e^x}+c$		
2	**a**	$4\log_e x+c, x>0$	**b**	$-3\log_e x+c, x>0$	**c**	$\frac{2}{5}\log_e x+c, x>0$		
	d	$\log_e(x+1)+c, x>-1$	**e**	$\frac{1}{2}\log_e x+c, x>0$	**f**	$x-2\log_e x-\frac{1}{x}+c, x>0$		
	g	$\frac{1}{2}x^2-2x+\log_e x+\mathrm{c}, x>0$	**h**	$3\ln(x+2)+c$				
3	**a**	$-\frac{1}{3}\cos(3x)+c$	**b**	$\frac{1}{2}\sin(2x)+c$	**c**	$\frac{1}{5}\tan(5x)+c$	**d**	$\cos(x)+\mathrm{c}$
4	**a**	$-\frac{1}{2}\cos(2x)+\frac{1}{2}x^2+c$	**b**	$2x^3-\frac{1}{4}\sin(4x)+\mathrm{c}$	**c**	$\frac{1}{5}e^{5x}+c$		

d $-\frac{4}{3}e^{-3x} - 2\cos\left(\frac{1}{2}x\right) + c$

e $3\sin\left(\frac{x}{3}\right) + \frac{1}{3}\cos(3x) + c$

f $\frac{1}{2}e^{2x} + 4\log_e x - x + \mathrm{c}, x > 0$

g $\frac{1}{2}e^{2x} + 2e^{x} + x + c$

h $\frac{5}{4}\cos(4x) + x - \log_e x + c, x > 0$

i $\frac{1}{3}\tan(3x) - 2\log_e x + 2e^{x/2} + c, x > 0$

j $\frac{1}{2}e^{2x} - 2x - \frac{1}{2}e^{-2x} + c$

k $\frac{1}{2}e^{2x+3} + c$

l $-\frac{1}{2}\cos(2x + \pi) + c$

m $\sin(x - \pi) + c$

n $-4\cos\left(\frac{1}{4}x + \frac{\pi}{2}\right) + c$

o $2\left(\frac{e^x + 2}{\sqrt{e^x}}\right) + c$

5 **a** $\frac{1}{16}(4x-1)^4 + c$

b $\frac{1}{21}(3x+5)^7 + c$

c $-\frac{1}{5}(2-x)^5 + c$

d $\frac{1}{12}(2x+3)^6 + c$

e $-\frac{1}{27}(7-3x)^9 + c$

f $\frac{1}{5}\left(\frac{1}{2}x - 2\right)^{10} + c$

g $-\frac{1}{25}(5x+2)^{-5} + c$

h $\frac{1}{4}(9-4x)^{-1} + c$

i $-\frac{1}{2}(x+3)^{-2} + c$

j $\ln(x+1) + c, x > -1$

k $\ln(2x+1) + c, x > -\frac{1}{2}$

l $-2\ln(3-2x) + c, x < \frac{3}{2}$

m $3\ln(5-x) + c, x < 5$

n $-\frac{3}{2}\ln(3-6x) + c, x < \frac{1}{2}$

o $\frac{5}{3}\ln(3x+2) + c, x > -\frac{2}{3}$

6 **a** $-\frac{1}{2}\cos(2x-3) - x^2 + c$

b $6\sin\left(2 + \frac{1}{2}x\right) + 5x + c$

c $\frac{3}{2}\sin\left(\frac{1}{3}x - 2\right) + \ln(2x+1) + c$

d $10\tan(0.1x - 5) - 2x + c$

e $2\ln(2x+3) + 2e^{-\frac{1}{2}x+2} + c$

f $-\frac{2}{2x+3} - \frac{1}{2}e^{2x-\frac{1}{2}} + c$

g $x + \ln(x+1) - 4\ln(x+2) + c$

h $2x - 3\ln(x+2) + \frac{1}{2}\ln(2x+1) + c$

i $-\frac{1}{2x+1} + \ln(2x+1) + c$

7 **a** $f(x) = \frac{1}{6}\sqrt{(4x+5)^3}$ **b** $f(x) = 2\ln(4x-3)+2$

c $f(x) = \frac{1}{2}\sin(2x+3)+1$ **d** $f(x) = 2x+\frac{1}{2}e^{-2x+1}+\frac{1}{2}e$

8 14 334

9 13.19ms^{-1} or 1.19ms^{-1}

10 2.66 cm

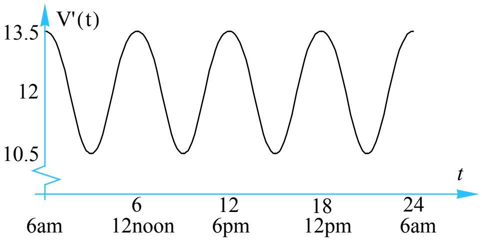
11 $2e^{x/2}-\frac{1}{2}\sin(2x)-2$

12 **a** $p = \frac{a}{a^2+b^2}$, $q = -\frac{b}{a^2+b^2}$ **b** $\frac{1}{13}e^{2x}(2\sin 3x-3\cos 3x)+c$

13 **a** $0.25a$ **b** $a\times\left(\frac{1}{2}\right)^{8/3}\approx 0.1575a$

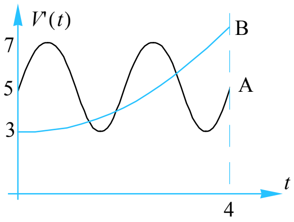
14 **b** 666 g

15 **a**



b 73.23% **c** ~25.24 litres

16 **a**



b 7000 **c** 1.16 day **d** 2 days

Exercise E.12.3

1 **a** $\frac{2}{3}(x^2+1)^{3/2}+c$ **b** $\frac{2}{3}(x^3+1)^{3/2}+c$ **c** $-\frac{1}{3}(4-x^4)^{1.5}+c$

d $\ln(x^3+1)+c$ **e** $-\frac{1}{18(3x^2+9)^3}+c$ **f** $e^{(x^2+4)}+c$

g $\ln(z^2+4z-5)+c$ **h** $-\frac{3}{8}(2-t^2)^{4/3}+c$ **i** $e^{\sin x}+c$

j $\ln[e^x+1]+c$ **k** $\frac{1}{5}\sin^5 x+c$ **l** $\frac{2}{5}(x+1)^{5/2}-\frac{2}{3}(x+1)^{3/2}+c$

2 **a** $\frac{1}{10}(2x-1)^{5/2}+\frac{1}{6}(2x-1)^{3/2}+c$ **b** $-\frac{2}{3}(1-x)^{3/2}+\frac{4}{5}(1-x)^{5/2}-\frac{2}{7}(1-x)^{7/2}+c$

c $\frac{2}{5}(x-1)^{5/2}+\frac{4}{3}(x-1)^{3/2}+c$ **d** $e^{\tan x}+c$ **e** $-\ln(1-2x^2)+c$

f` $\frac{1}{1-2x^2}+c$ **g** $\frac{1}{2}(\ln x)^2+c$ **h** $-\ln(1+e^{-x})+c$

i $\ln(\ln x)+c$

3 **a** 0 **b** $\frac{2\ln 2}{3}$ **c** $\ln\frac{77}{54}$

d $\ln 2$ **e** $\frac{1}{3}\ln 2$ **f** $\frac{1}{4}$

g $\frac{76}{15}$ **h** $\frac{16}{15}$ **i** $\frac{2}{3}(1+e)^{3/2}(1-e^{-3/2})$

4 **a** $\frac{7\sqrt{7}}{3}-\frac{8}{3}$ **b** $\frac{3}{8}(\cos\pi^2-1)$ **c** $\frac{1042}{5}$

d $\ln 4$ **e** 1 **f** $\frac{5}{4}(e^5-e^{-1})$

g 24 414 **h** $\sqrt{3}-\sqrt{2}$ **i** $\frac{1}{4}\ln 3$

5 **a** $\frac{1}{4}$ **b** $2-\frac{2}{3}\sqrt{3}$ **c** $\frac{31}{80}$

d $4-2\sqrt{2}$ **e** $\ln 2$ **f** $\frac{2}{3}$

6 **a** $-\frac{2}{5}\sqrt{3}$ **b** $\frac{2}{5}\sqrt{3}$ **c** $\frac{26}{3}$

d $-\frac{4}{3}$ **e** $-\frac{56}{15}\sqrt{2}$ **f** $3 + 2\ln 4$

7 **a** $\text{Tan}^{-1}(x+3)+c$ **b** $\frac{2}{\sqrt{3}}\text{Tan}^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+c$ **c** $\text{Sin}^{-1}\left(\frac{x-2}{\sqrt{5}}\right)+c$

d $3\text{Sin}^{-1}\left(\frac{x+1}{3}\right)+c$ **e** $2\sin^{-1}\left(\frac{2x-3}{\sqrt{29}}\right)+c$ **f** $\frac{1}{2}\sin^{-1}\left(\frac{x^2}{3}\right)+c$

g $\frac{1}{2}(\arcsin x)^2+c$ **h** $-\frac{1}{3}(\arccos x)^3+c$ **i** $-\frac{1}{2}(\arcsin x)^{-2}+c$

8 **a** $A = 1, B = -2$

9 **a** $\text{Tan}^{-1}k$ **b i** $\frac{\pi}{6}$ **ii** $\frac{\pi}{4}$ **c** $\frac{\pi}{2}, \pi$

10 $2\sqrt{x}-2\ln(\sqrt{x}+1), 2-2\ln 2$

11 $\frac{3k^2\pi}{8}$

12 $\frac{\pi a^2}{4}$

13 **a** $\frac{\pi}{3}$ **b** $8\text{Sin}^{-1}\left(\frac{2}{3}\right)$ **c** $\frac{\pi}{4}$

d $\frac{1}{2}\text{Sin}^{-1}(1)$ **e** $2\sqrt{2}-2-\frac{\pi}{2}$ **f** $\frac{\pi}{4}$

g $\pi-2\text{Tan}^{-1}\left(\frac{1}{3}\right)$

Exercise E.12.4

1 **a** x^2+x+3 **b** $2x-\frac{1}{3}x^3+1$ **c** $\frac{8}{3}\sqrt{x^3}-\frac{1}{2}x^2-\frac{40}{3}$

d $\frac{1}{2}x^2+\frac{1}{x}+2x-\frac{3}{2}$ **e** $(x+2)^3$ **f** $\frac{3}{4}\sqrt[3]{x^4}+\frac{1}{4}x^4+x$

g $\frac{1}{3}x^3+1$ **h** x^4-x^3+2x+3

2 $\frac{1}{2}x^2+\frac{1}{x}+\frac{5}{2}$

3 \$3835.03

4 9.5

5 $\frac{251}{3}\pi$ cm^3

6 292

7 $\frac{5}{7}\sqrt{x^3}+\frac{23}{7}$

8 1, −8

9 $P(x) = 25-5x+\frac{1}{3}x^2$

10 $N = \frac{20000}{201}t^{2.01}+500, t \geq 0$

11 **a** $y = -\frac{2}{5}x^2+4x$ **b** $y = \frac{1}{6}x^3+\frac{5}{4}x^2+2x$

12 $y = 2(x^3+x^2+x)$

13 $f(x) = -\frac{3}{10}x^3+\frac{49}{10}x-\frac{13}{5}$

14 Vol ~ 43 202 cm^3

15 110 cm^2

Exercise E.12.5

1 **a** $\frac{15}{2}$ **b** $\frac{38}{3}$ **c** $\frac{5}{36}$ **d** −8

2 **a** $\frac{35}{24}$ **b** $\frac{8}{5}\sqrt{2}-2$ **c** −2 **d** 0

e $\frac{1}{20}$ **f** $-\frac{4}{3}$ **g** $\frac{7}{6}$ **h** $\frac{5}{6}$

i $\frac{20}{3}$ **j** 0 **k** $\frac{20}{3}$ **l** $-\frac{\sqrt{2}}{3}$

4 **a** e **b** $2(e^{-2}-e^{-4})$ **c** 0 **d** $2(e-e^{-1})$

e e^2+4-e^{-2} **f** $\frac{1}{2}(e-e^5)$ **g** $2\sqrt{e}-3$

h $\frac{1}{4}(16e^{1/4}-e^4-15)$ **i** $\frac{1}{2}(e^{-1}-e^3)$

6 **a** $3\ln 2$ **b** 2ln5 **c** 4 + 4ln3 **d** $\frac{1717}{4}$

e $\frac{3}{2}\ln 3$ **f** 2ln2 **g** $\frac{3}{4}$ **h** 4ln2 − 2 **i** ln2

8 **a** 1 **b** $\frac{3\sqrt{3}}{2}$ **c** $\frac{\sqrt{3}}{2}$ **d** -2

e $\frac{\pi^2}{32}-1$ **f** 0 **g** 0 **h** $\frac{\sqrt{3}}{2}-\frac{1}{2}$

i 0 **j** 2

9 **a** $\frac{31}{5}$ **b** $\frac{7\sqrt{7}}{3}-\sqrt{3}$ **c** 0 **d** $\frac{5}{72}$

e $3\sqrt[3]{2}-\frac{3}{2}$ **f** $1-\ln 2$ **g** $\frac{76}{15}$ **h** $\frac{16}{15}$

i $\frac{2}{3}(e+1)^{3/2}(1-e^{-3/2})$

10 $\ln\left(\frac{21}{5}\right)$

11 $\sin 2x + 2x\cos 2x$; 0

12 **a** $2m-n$ **b** $m+a-\mathrm{b}$ **c** $-3n$ **d** $m(2a-b)$ **e** na^2

13 **a** $e^{0.1x}+0.1xe^{0.1x}$; $10xe^{0.1x}-100e^{0.1x}+c$

b i 99 accidents **ii** $N = 12t+10te^{0.1t}-100e^{0.1t}+978$

14 **a** 1612 subscribers **b** 46 220

15 **b** ~524 flies

Exercise E.12.6

1 **a** 4 sq.units **b** $\frac{32}{3}$ sq.units **c** 4 sq.units

d 36 sq.units **e** $\frac{1}{6}$ sq.units

2 **a** e sq.units **b** $\frac{1}{2}(e^4-2-e^2)$ sq.units **c** $2(e+e^{-1}-2)$ sq.units

d $2(e^2-2-e)$ sq.units

3 **a** $\ln\left(\frac{5}{4}\right)$ sq.units **b** 2ln5 sq.units **c** 3ln3 sq.units **d** 0.5 sq.units

4 **a** 2 sq.units **b** $\frac{\pi}{2}$ sq.units **c** $\frac{3}{8}\pi^2+\sqrt{2}-2$ sq.units

d $\sqrt{2}$ sq. units **e** $4\sqrt{3}$ sq.units

6 12 sq. units

7 $4\left(\sqrt{3}-\frac{1}{3}\right)$ sq.units.

8 ln2 + 1.5 sq.units.

9 2 sq.units.

10 $\frac{37}{12}$ sq. units

11 **a** 0.5 sq. units **b** 1 sq. unit **c** $2(\sqrt{6}-\sqrt{2})$ sq. units

12 $\frac{8}{3}$

13 $-2\tan 2x$; $\frac{1}{4}\ln 2$ sq.units

14 **a** $\frac{9}{2}$ sq. units **b** 3 sq. units

15 **a** 1 sq.unit **b** 10 sq. units

16 **a** $x\ln x - x + c$ **b** 1 sq. unit

17 $\frac{14}{3}$ sq. units

18 **a** $\frac{7}{6}$ sq. units **b** $\frac{9}{2}$ sq. units

19 **a i** $\frac{15}{4}$ sq. units **ii** $\frac{45}{4}$ sq. units

20 $\frac{22}{3}$ sq. units

21 **b i** $e^{-1}+e-2$ sq. units **ii** 1 sq. unit **iii** 2ln(2) sq. units

22 **b** 3.05 sq. units

23 **a** $2y = 3ax - a^3$ **b** $\frac{1}{15}a^5$ sq. units

24 **a** $1-e^{-1}$ sq. units **b** e^{-1} sq. units **c** $1-e^{-e^{-1}-1}-e^{-1}$ ~ 0.10066 sq. units

Exercise E.12.7

All values are in cubic units.

1 21π

2 $p\ln 5$

3 $\frac{4}{5}\pi$

4 $\frac{\pi}{2}(e^{10}-e^{2})$

5 π^2

6 $\frac{\pi}{2}$

7 $\frac{109}{3}\pi$

8 $\pi\left(\frac{8}{3}-2\ln 3\right)$

12 $\frac{\pi}{2}(5-5\sin 1)$

13 $\frac{251}{30}\pi$

14 **a** 40π **b** $\frac{242}{5}\pi$

15 **a** $\frac{8}{35}\pi$ **b** $\frac{\pi}{4}$

16 **a** $\frac{9}{2}\pi$ **b** $\frac{88}{5}\sqrt{3}\pi$

17 $\frac{3\pi}{4}$

18 $k = 1$

19 $4\pi^2 a^2$

20 $k = \frac{\pi}{2}$

21 **b i** $\frac{\pi a}{2(1+a^2)}$ **ii** $\frac{8\pi}{15}\sqrt{\frac{a}{1+a^2}}\left(\frac{3a^2+2}{1+a^2}\right)$

22 **a** Two possible solutions: solving $a^3-6a^2-36a+204 = 0$, $a = 4.95331$; solving $a^3-6a^2-36a-28 = 0$, then $a = -0.95331$

b $a = \frac{100}{\pi}$

23 $\frac{28}{15}\pi$

24. **a** $\frac{1472}{15}\pi$ **b** 64π **c** $\frac{576}{5}\pi$

Exercise E.13.1

1. a 4 m b 2 m/s c 2 m/s^2 d 6 m/s

2. a $h'(t) = 0.5e^{0.5t}$, $h''(t) = 0.25e^{0.5t}$ b 74 m/s

3. a $h'(t)=\dfrac{4x-1}{2\sqrt{2t^2-t+1}}, 0\le t\le 5$ and $h''(t)=\dfrac{7}{4\sqrt{2t^2-t+1}\left(2t^2-t+1\right)}, 0\le t\le 5$

b At $t = 5$ speed ≈ 1.87

4. a $d'(t)=\dfrac{-1}{(2t+1)^2}$ and $d''(t)=\dfrac{4}{(2t+1)^3}$ b –0.25 m/min

5. $v = -8$, $a = -2$

Exercise E.13.2

1 **a** $x = t^3 + 3t + 10, t \ge 0$ **b** $x = 4\sin t + 3\cos t - 1, t \ge 0$ **c** $x = t^2 - 4e^{-\frac{1}{2}t} + 2t + 4, t \ge 0$

2 **a** $x = t^3 - t^2, t \ge 0$ **b** 100 **c** $100\frac{8}{27}$ m

3 **a** $x = -\frac{2}{3}(4 + t)^{3/2} + 2t + 8$ **b** 6.92 m

4 $\frac{125}{6}$ m

5 $\frac{125}{49}$ s; 63.8 m

6 **a** $\frac{\pi}{6}$ s **b** $\frac{\pi}{2} - 1$ m

7 80.37 m

8 **a** $s(t) = \dfrac{160}{\pi}\left[1 - \cos\left(\dfrac{\pi}{16}t\right)\right], t \ge 0$ **b** 86.94 m

c –6.33 m **d** 116.78 m

9 **a** $v = 4 + k - \dfrac{k}{t^2}, t > 0$ **b** $k = 2$ **c** 52.2 m

10 **b** 0.0893 m

Exercise E.14.1

1. The rate of change of y with respect to x is proportional to the quotient of x on y. $\frac{dy}{dx}=x+y$

2. P = pressure, t = time. $\frac{dP}{dt}=k$

3. V = volume, t = time. $\frac{dV}{dt}=-0.01$

4. V = volume, t = time. $\frac{dV}{dt}=kr^3$

5. V = volume, t = time, d = depth $V(0) = 500$, $d(0) = 200$

   a $\frac{dV}{dt}=-kd$

   b $\frac{dV}{dt}=-k(d-100)$

6. C = concentration, t = time. $\frac{dC}{dt}=-kC$

7. R = radius, A = area, t = time. $\frac{dr}{dt}=-\frac{k}{r^2}$

8. R = radius, A = area, t = time. $\frac{dA}{dr}=kr$

9. A = amount, t = time $A(0) = 3000$. $\frac{dA}{dt}=1.06$

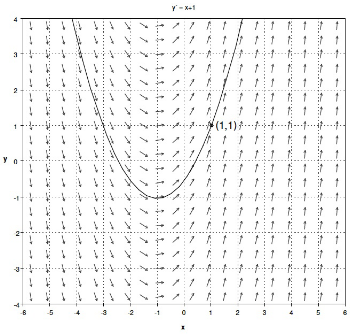
10. P = pressure, t = time minutes $P(0) = 110$. $\frac{dP}{dt}=1.05$

11. T = temperature of object at time t. T_0 is temperature of the surroundings. $\frac{dT}{dt}=-k(T-T_0)$

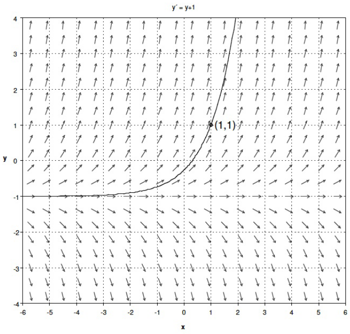
12. T = temperature of object at time t, energy = E. $\frac{dE}{dt}=kT^4$

Exercise E.14.2

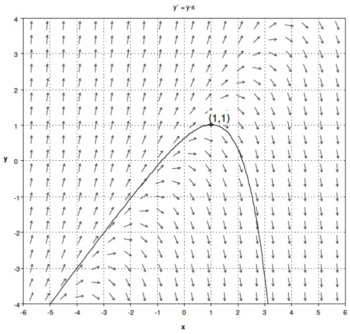
1&2. a



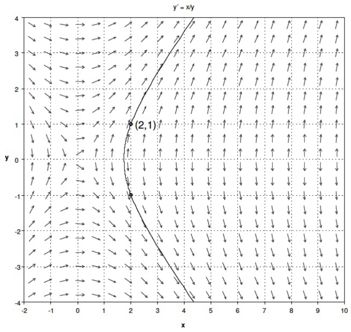
c



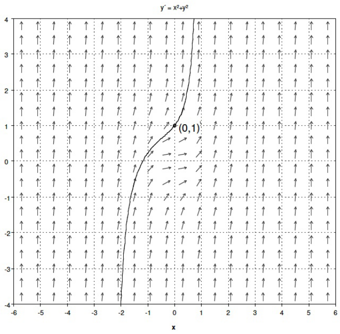
d



3.

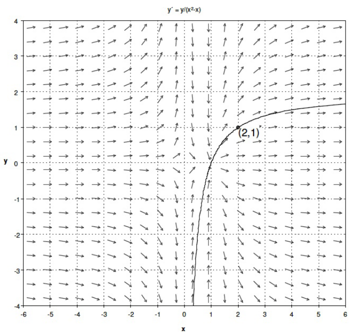


4.



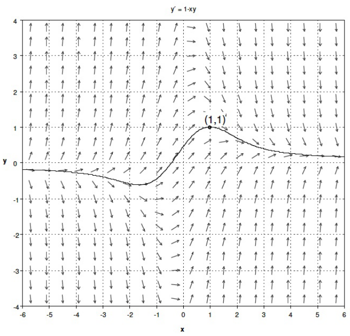
x - 1, y≈ 2

5. a

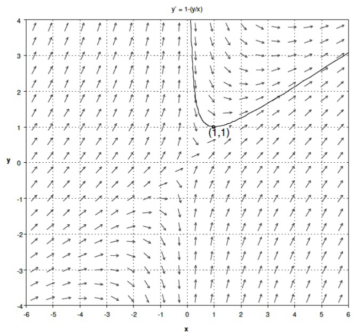


(b) and (c) The solution curve passes through the point (2, 1).

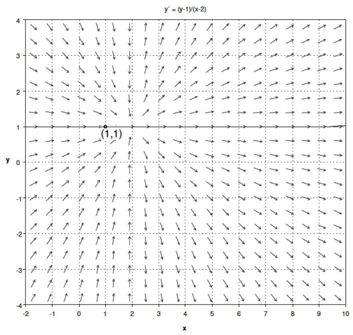
6. a


y´ = 1-xy
(1,1)
y
x

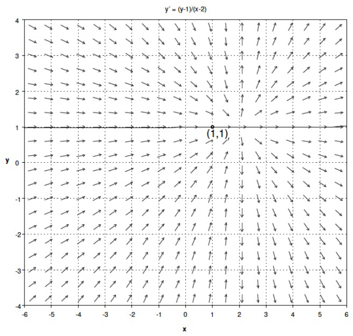
b


y´ = 1-(y/x)
(1,1)
y
x

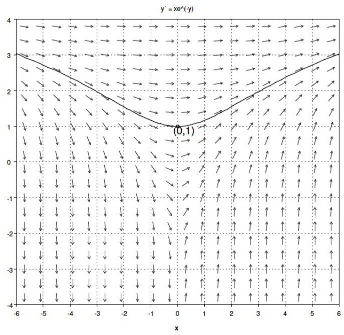
7. a


y´ = (y-1)/(x-2)
(1,1)
y
x

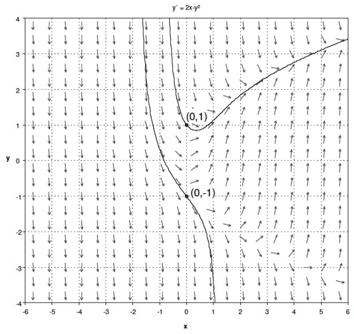
b


y´ = (y-1)/(x-2)
(1,1)
y
x

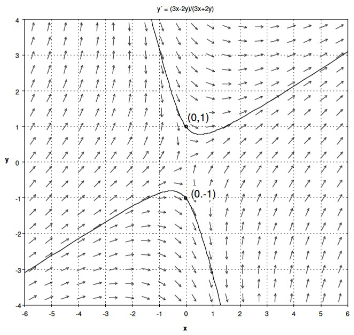
8.


y´ = xe^(-y)
(0,1)
x

9.


y´ = 2x-y²
(0,1)
(0,-1)
y
x

10.


y´ = (3x-2y)/(3x+2y)
(0,1)
(0,-1)
y
x

Exercise E.14.3

1. a $kx = e^{y/x}$ b $kx = e^{-y/x}$ c $y^2 = kx^3 + x^2$

d $\log_e(kx) = -e^{-y/x}$ e $x = kx^2(x-2y)$ f $x^2 - 2xy - y^2 = k$

2. $x^2 + y^2 = 5x^3$

3. $y = -\frac{e^{-2x}}{2} + 2x + \frac{e^{-2}}{2}$

4. a $C = \frac{2}{2+t}$ b 0.4 mole/l c 0.857 hours.

5.

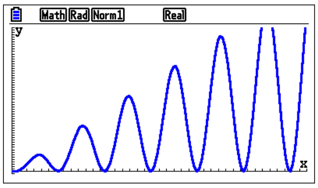
a $h_{\max} = 127.3$ cm

b
$$\frac{dV}{dt} = -kh$$
$$\Rightarrow 200 = -k \times 80$$
$$\Rightarrow k = 2.5$$
$$\Rightarrow \frac{dV}{dt} = -2.5h$$

c
$$V = \pi \times 50^2 h \Rightarrow \frac{dV}{dh} = 2500\pi$$
$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV} = \frac{-2.5h}{2500\pi} = \frac{-h}{1000\pi}$$

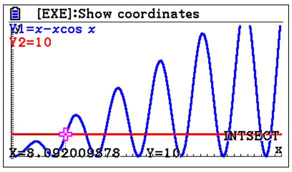
d
$$\frac{dh}{dt} = \frac{-h}{1000\pi}$$
$$\int \frac{dh}{h} = -\frac{1}{1000\pi}\int dt$$
$$\log_e h = \frac{-h}{1000\pi} + c$$
$$h = Ae^{-t/1000\pi} = 127.32e^{-t/1000\pi}$$
$$t = 200, h \approx 119.47, V \approx 838l$$

6. a $E = t - t\cos t$ b



c 18.4

d


[EXE]:Show coordinates

Y1=x−xcos x

Y2=10

X=8.092009878 Y=10

~8.1 sec

Exercise E.14.4

1. a ~ 4.0625 b ~ 5.1917

c $h = 0.5$; error = 36.41%. $h = 0.2$; error = 18.74%

2. a i. ~ 25.27 ii. ~ 18.235

b $h = 0.1$; error = 37.09%. $h = 0.2$; error = 54.61%

3. ~ 0.648

4. ~ 7. ~ 2.2991×10^{29}

5. 190

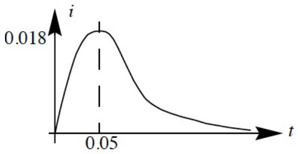
6. ~ 0.736

7. ~ 2.629

8. a i $S(t) = 80e^{-t10}, t \geq 0$ ii $S(t) = 100 - 20e^{-t10}, t \geq 0$

b i $S(t) = 80\left(\frac{100}{100+2t}\right)^5, t \geq 0$ ii $S(t) = \frac{5}{6}(100+2t) - \frac{10}{3}\left(\frac{100}{100+2t}\right)^5, t \geq 0$

9. a $i(t) = te^{-20t}, t \geq 0$ b



c i $0.05e^{-1} \approx 0.018$ ii $0.05e^{-1} \approx 0.018$

10.

$$\frac{dx}{dt} + \alpha x = \beta(t)$$

when $\beta(t) = \beta$ and $\beta \in \mathbb{R}$ we have

$$\frac{dx}{dt} + \alpha x = \beta$$

$$I(t) = e^{\int \alpha \, dt} = e^{\alpha t}.$$

So $\frac{d}{dt}\left(e^{\alpha t} x\right) = \beta e^{\alpha t}$

Integrate b.s.w.r.t. t: $e^{\alpha t} x = \frac{\beta}{\alpha} e^{\alpha t} + c$

So $x = \frac{\beta}{\alpha} + ce^{-\alpha t}$.

Logically, when $t = 0$, $x = 0$, so

$$c = -\frac{\beta}{\alpha}$$

So $x = \frac{\beta}{\alpha} - \frac{\beta}{\alpha} e^{-\alpha t}$

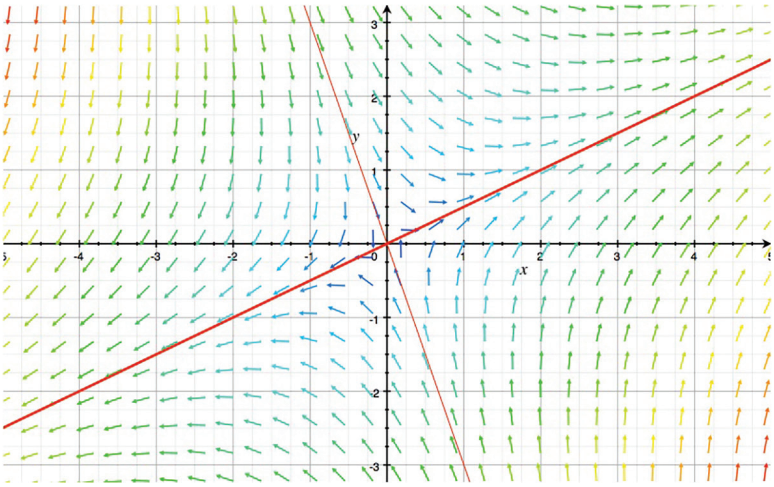
$\Rightarrow x = \frac{\beta}{\alpha}\left(1 - e^{-\alpha t}\right)$.

Now, as time increases, $e^{-\alpha t} \to 0$, and $x \to \frac{\beta}{\alpha}$.

The height of a tree approaches $\frac{\beta}{\alpha}$ as time increases. The height will remain under β/α.

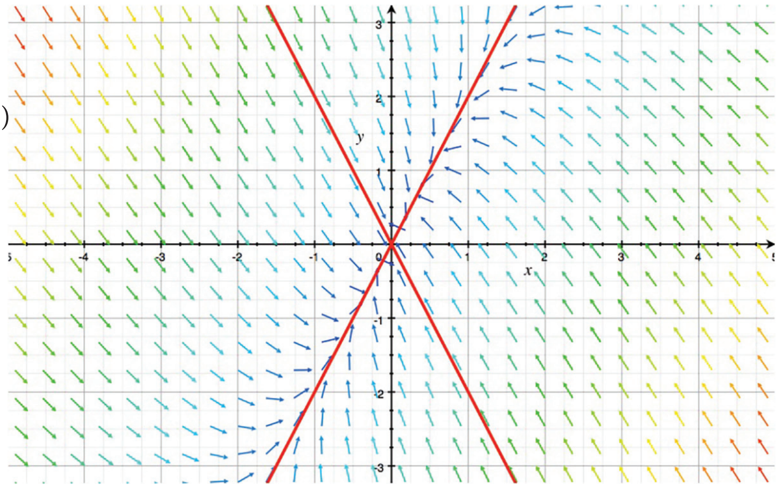
Exercise E.14.5

1. a $k=-4,3, y=\frac{x}{2}, y=-3x$

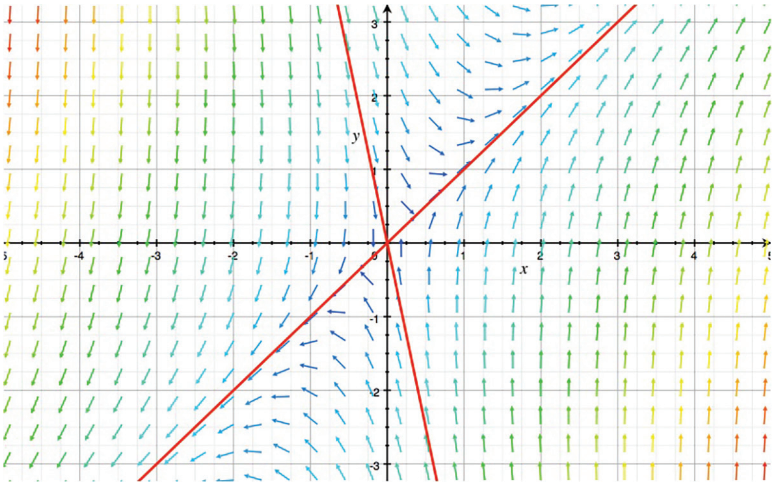


b $k=-5,-1, y=-2x, y=2x$

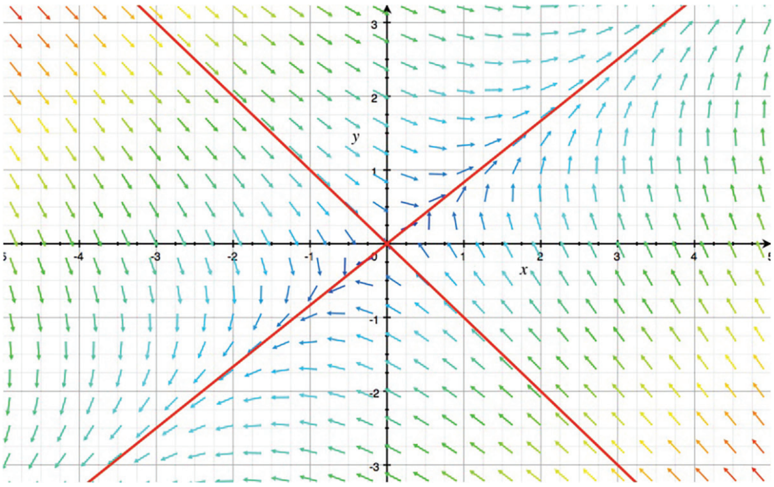
Solutions on $y=-2x$ decay to (0,0)



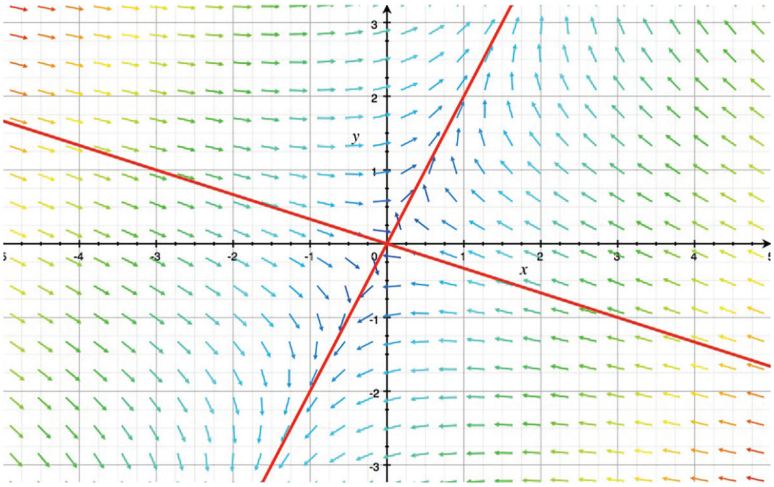
c $k=-4,2, y=-\frac{1}{5}x, y=x$



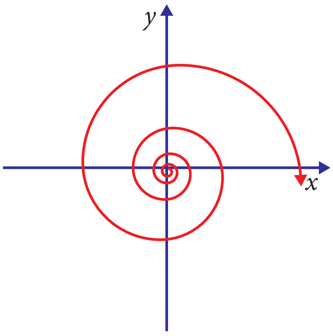
d $k=-8,3, y=-x, y=\frac{5}{6}x$



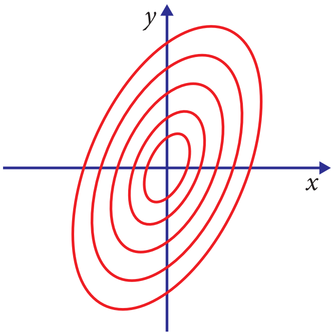
e e $k=-5,2,\ y=2x,\ y=-\frac{1}{3}x$



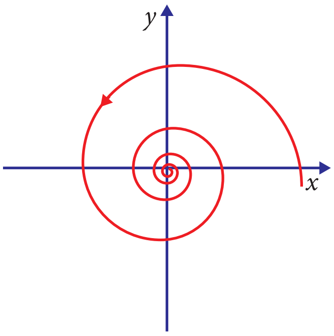
2. a



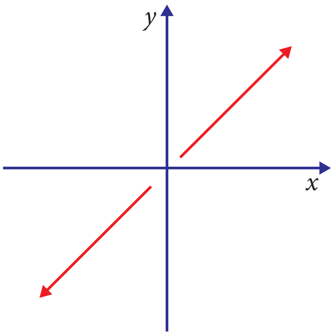
b



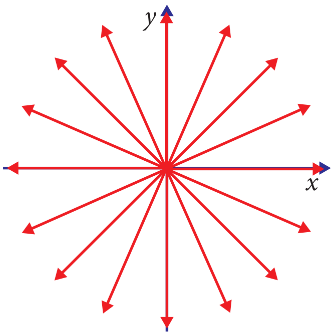
c



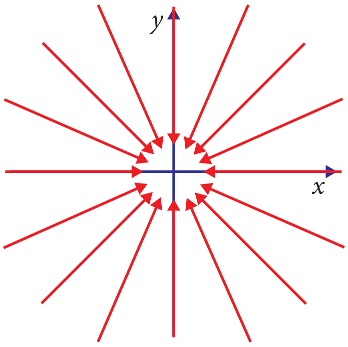
d



e



f

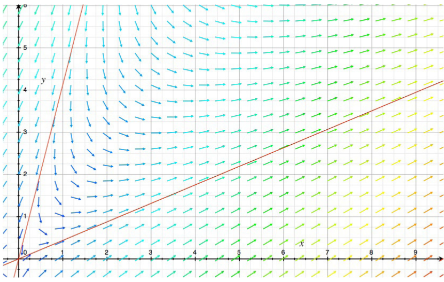


g Saddle point at the origin.

Exercise E.14.6

1. a $k=\frac{1\pm\sqrt{17}}{2}$

b



Aphids horizontal.

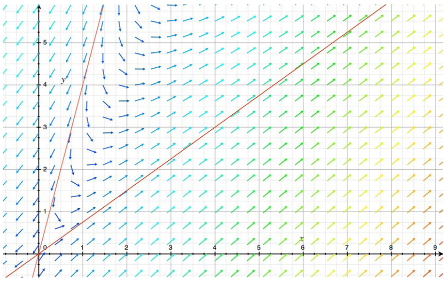
c Start with more than 5:1 L to A populations degenerate to zero. All others move to stable increasing.

d Stable is ~0.483 ladybugs per aphid.

2. a $\frac{dP_A}{dt}=1.3P_A(t)\Rightarrow P_A(t)=e^{1.3t}$ doubles in ~5.3 years $\frac{dP_B}{dt}=1.2P_B(t)\Rightarrow P_B(t)=e^{1.2t}$ doubles in ~5.8 years.

b −0.3, 1

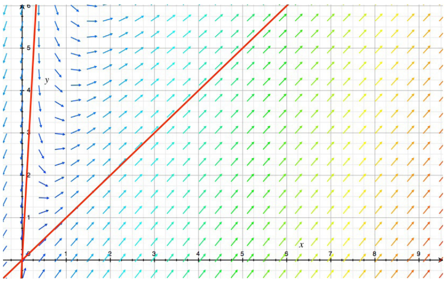
c



d 4:3 ratio in favour of A.

3. a −0.4, 1.4

b



c,d All but the top left tend to a 1:1 ratio

4. a $S'=1.1S$

$P'=1.05P$

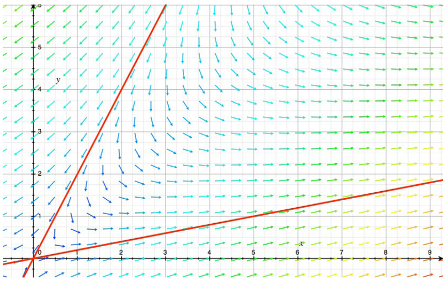
b $S'=1.1S-0.2P$

$P'=1.05P-0.4S$

c The eigenvalues are complex but with a positive real part. The reef will grow.

5. a $k_1=12,\begin{bmatrix}5\\1\end{bmatrix},k_2=-6,\begin{bmatrix}1\\2\end{bmatrix}$

d



c $X(t)=5Ae^{12t}+Be^{-6t}$

$Y(t)=Ae^{12t}+2Be^{-6t}$

d $X'(t)=60Ae^{12t}-6Be^{-6t}$

$Y'(t)=12Ae^{12t}-12Be^{-6t}$

e
$$\begin{aligned}14X-10Y&=14\left(5Ae^{12t}+Be^{-6t}\right)-10\left(Ae^{12t}+2Be^{-6t}\right)\\&=70Ae^{12t}+14Be^{-6t}-10Ae^{12t}-20Be^{-6t}\\&=60Ae^{12t}-6Be^{-6t}\\&=X'\\4X-8Y&=4\left(5Ae^{12t}+Be^{-6t}\right)-8\left(Ae^{12t}+2Be^{-6t}\right)\\&=20Ae^{12t}+4Be^{-6t}-8Ae^{12t}-16Be^{-6t}\\&=12Ae^{12t}-12Be^{-6t}\\&=Y'\end{aligned}$$

f
$$\begin{aligned}X(t)&=5Ae^{12t}+Be^{-6t}\\Y(t)&=Ae^{12t}+2Be^{-6t}\end{aligned}$$

6. The concentrations follow an elliptical paths with a pattern that repeats itself.

7. a Saddle point at the origin

b Stable critical point at the origin.

c Unstable critical point at the origin.

Exercise E.14.7

1. a $y=\frac{x^3}{3}+\frac{x^2}{2}+x+1$ b $y=\frac{x^4}{12}+\frac{x^2}{2}+x+1$ c $y=\frac{x}{2}-\frac{1}{4}\sin 2x$

d $y=e^x+\frac{x^3}{6}-1$ e $y=\frac{e^x-e^{-x}}{2}$

2. a
$$\begin{aligned}&s''=3\\&s'=3t+c_1, t=0, s'=0\Rightarrow s'=3t\\&s=\frac{3t^2}{2}+c_2, t=0, s=0\Rightarrow s=\frac{3t^2}{2}\end{aligned}$$
b 150m

3. a 20sec b 2 km

4. Yes.

6. 44.7 m

8. $A=\frac{t^3}{2}-2t+10$

9. a $V_{\text{horiz.}}=v.\cos 15^\circ,\ V_{\text{vert.}}=v.\sin 15^\circ$

b $V'_{\text{vert.}}=-10\Rightarrow V_{\text{vert.}}=-10t+v.\sin 15^\circ$

c $$V_{\text{vert.}}=-10t+v.\sin 15^{\circ}=0\Rightarrow t=\frac{v.\sin 15^{\circ}}{10}$$

d $$\frac{v.\sin 15^{\circ}}{5}$$

e $$v^{2}=\frac{5\times 3340}{\sin 15^{\circ}\times \cos 15^{\circ}}\Rightarrow v\approx 258\text{ ms}^{-2}$$

10. $$y=Ae^{3x}+\frac{2x}{3}+B$$

11. 26° or 75°

Exercise A.7.1

1 Simplify the following.

e $\left(\frac{2x^3}{4y^2}\right)^2 \times \frac{12y^6}{8x^4}$

f $\frac{3^{n+2}+9}{3}$

g $\frac{4^{n+2}-16}{4}$

h $\frac{4^{n+2}-16}{2}$

i $\left(\frac{1}{2b}\right)^4 - \frac{b^2}{16}$

2 Simplify the following.

e $\frac{(xy)^6}{64x^6}$

f $\frac{27^{n+2}}{6^{n+2}}$

3 Simplify the following.

e $\frac{2^n \times 4^{2n+1}}{2^{1-n}}$

f $\frac{2^{2n+1} \times 4^{-n}}{(2^n)^3}$

g $\frac{x^{4n+1}}{(x^{n+1})^{(n-1)}}$

h $\frac{x^{4n^2+n}}{(x^{n+1})^{(n-1)}}$

i $\frac{(3^x)(3^{x+1})(3^2)}{(3^x)^2}$

5 Simplify the following, leaving your answer in positive power form.

e $\frac{(-2)^3 \times 2^{-3}}{(x^{-1})^2 \times x^2}$

f $\frac{(-a)^3 \times a^{-3}}{(b^{-1})^{-2}b^{-3}}$

6 Simplify the following.

e $\frac{(x-1)^{-3}}{(x+1)^{-1}(x^2-1)^2}$

f $\frac{y(x^{-1})^2 + x^{-1}}{x+y}$

7 Simplify the following.

a $5^{n+1} - 5^{n-1} - 2 \times 5^{n-2}$

b $a^{x-y} \times a^{y-z} \times a^{z-x}$

c $\left(\frac{a^{-\frac{1}{2}}b^3}{ab^{-1}}\right)^2 \times \frac{1}{ab}$

d $\left(\frac{a^{m+n}}{a^n}\right)^m \times \left(\frac{a^{n-m}}{a^n}\right)^{m-n}$

e $\frac{p^{-2}-q^{-2}}{p^{-1}-q^{-1}}$

f $\frac{1}{1+a^{\frac{1}{2}}} - \frac{1}{1-a^{\frac{1}{2}}}$

g $\frac{2^{n+4}-2(2^n)}{2(2^{n+3})}$

h $\sqrt{a\sqrt{a\sqrt{a}}}$

8 Simplify the following.

a $\frac{\sqrt{x} \times \sqrt[3]{x^2}}{\sqrt[4]{x}}$

b $\frac{b^{n+1} \times 8a^{2n-1}}{(2b)^2(ab)^{-n+1}}$

c $\frac{2^n - 6^n}{1-3^n}$

d $\frac{7^{m+1}-7^m}{7^n-7^{n+2}}$

e $\frac{5^{2n+1}+25^n}{5^{2n}+5^{1+n}}$

Exercise A.7.2

1. Solve the following equations.

g $\{x|\ 3^{2x-4} = 1\}$

h $\{x|\ 4^{2x+1} = 128\}$

i $\{x|\ 27^{x} = 3\}$

Exercise A.7.3

1 Use the definition of a logarithm to determine the following.

g $\log_4 1$ h $\log_{10} 1$ i $\log_{\frac{1}{2}} 2$ j $\log_{\frac{1}{3}} 9$

k $\log_3 \sqrt{3}$ l $\log_{10} 0.01$

3 Change the following logarithmic expression into its equivalent exponential form.

f $\log_2(ax - b) = y$

4 Solve for x in each of the following.

g $\log_x 16 = 2$ h $\log_x 81 = 2$

i $\log_x\left(\frac{1}{3}\right) = 3$ j $\log_2(x-5) = 4$

k $\log_3 81 = x + 1$ l $\log_3(x-4) = 2$

5 Solve for x in each of the following, giving your answer to 4 d.p.

g $\log_e(x+2) = 4$ h $\log_e(x-2) = 1$ i $\log_x e = -2$

Exercise A.7.4

1 Without using a calculator, evaluate the following.

e $\log_2 20 - \log_2 5$ f $\log_2 10 - \log_2 5$

2 Write down an expression for $\log a$ in terms of $\log b$ and $\log c$ for the following.

e $a = b^3c^4$ f $a = \frac{b^2}{\sqrt{c}}$

4 Express each of the following as an equation that does not involve a logarithm.

d $\log_2 x = y + 1$ e $\log_2 y = \frac{1}{2}\log_2 x$ f $3\log_2(x+1) = 2\log_2 y$

5 Solve the following equations.

d $\log_{10}(x+3) - \log_{10}x = \log_{10}x + \log_{10}2$

e $\log_{10}(x^2+1) - 2\log_{10}x = 1$

f $\log_2(3x^2+28) - \log_2(3x-2) = 1$

g $\log_{10}(x^2+1) = 1 + \log_{10}(x-2)$

h $\log_2(x+3) = 1 - \log_2(x-2)$

i $\log_6(x+5) + \log_6 x = 2$

j $\log_3(x-2) + \log_3(x-4) = 2$

k $\log_2 x - \log_2(x-1) = 3\log_2 4$

l $\log_{10}(x+2) - \log_{10}x = 2\log_{10}4$

6 Simplify the following

c $2\log_a x + 3\log_a(x+1)$ d $5\log_a x - \frac{1}{2}\log_a(2x-3) + 3\log_a(x+1)$

e $\log_{10}x^3 + \frac{1}{3}\log x^3y^6 - 5\log_{10}x$ f $2\log_2 x - 4\log_2\left(\frac{1}{y}\right) - 3\log_2 xy$

7 Solve the following

d $\log_3 x + \log_3(x-8) = 2$

e $\log_2 x + \log_2 x^3 = 4$

f $\log_3\sqrt{x} + 3\log_3 x = 7$

8 Solve for *x*.

c $\log_4 x^4 = (\log_4 x)^4$ d $\log_5 x^5 = (\log_5 x)^5$

e Investigate the solution to $\log_n x^n = (\log_n x)^n$.

9 Solve the following, giving an exact answer and an answer to 2 d.p.

e $3^{4x+1} = 10$ f $0.8^{x-1} = 0.4$ g $10^{-2x} = 2$

h $2.7^{0.3x} = 9$ i $0.2^{-2x} = 20$ j $\frac{2}{1 + 0.4^x} = 5$

k $\frac{2^x}{1 - 2^x} = 3$ l $\frac{3^x}{3^x + 3} = \frac{1}{3}$

10 Solve for *x*.

c $\log_{10}(x^2 - 3x + 6) = 1$ d $(\log_{10} x)^2 - 11\log_{10} x + 10 = 0$

e $\log_x(3x^2 + 10x) = 3$ f $\log_{x+2}(3x^2 + 4x - 14) = 2$

11 Solve the following simultaneous equations.

c
$$\begin{aligned} xy &= 2 \\ 2\log_2 x - \log_2 y &= 2 \end{aligned}$$

12 Express each of the following as an equation that does not involve a logarithm.

c $\ln x = y - 1$

13 Solve the following for *x*.

c $\log_e(x + 1) + \log_e x = 0$ d $\log_e(x + 1) - \log_e x = 0$

14 Solve the following for *x*.

c $-5 + e^{-x} = 2$ d $200e^{-2x} = 50$ e $\frac{2}{1 - e^{-x}} = 3$

f $70e^{-\frac{1}{2}x} + 15 = 60$ g $\ln x = 3$ h $2\ln(3x) = 4$

i $\ln(x^2) = 9$ j $\ln x - \ln(x + 2) = 3$ k $\ln\sqrt{x + 4} = 1$ l $\ln(x^3) = 9$

15 Solve the following for x.

c $e^{2x} - 5e^{x} + 6 = 0$ d $e^{2x} - 2e^{x} + 1 = 0$

e $e^{2x} - 6e^{x} + 5 = 0$ f $e^{2x} - 9e^{x} - 10 = 0$

16 Solve each of the following.

a $4^{x-1} = 132$ b $5^{5x-1} = 3^{1-2x}$

c $3^{2x+1} - 7 \times 3^{x} + 4 = 0$ d $2^{2x+3} - 7 \times 2^{x+1} + 5 = 0$

e $3 \times 4^{2x+1} - 2 \times 4^{x+2} + 5 = 0$ f $3^{2x} - 3^{x+2} + 8 = 0$

g $2\log x + \log 4 = \log(9x - 2)$ h $2\log 2x - \log 4 = \log(2x - 1)$

i $\log_3 2x + \log_3 81 = 9$ j $\log_2 x + \log_x 2 = 2$

Exercise A.9.1

25. The equation $z+b+i(z-4) = 0$, where b is a real number, has as its solution a real number. Determine this solution and hence determine the value of b.

26. Express the following in the form $a+bi$ where a and b are real numbers.

a $\dfrac{\cos 2\theta + i\sin 2\theta}{\cos\theta + i\sin\theta}$

b $\dfrac{\cos\theta + i\sin\theta}{\cos 3\theta - i\sin 3\theta}$

27*. Let the complex matrix $A = \begin{bmatrix} \alpha i & 0 \\ 0 & -\beta i \end{bmatrix}$. Find:

a A^2 b A^4

c A^{-1} d A^{4n}, where n is a positive integer.

28*. Find $\dfrac{dy}{d\theta}$ given that $y = \cos\theta + i\sin\theta$.

Show: i $i \cdot \dfrac{d\theta}{dy} = \dfrac{1}{y}$ ii when $\theta = 0, y = 1$.

Hence show that $e^{i\theta} = \cos\theta + i\sin\theta$.

Deduce an expression for $e^{-i\theta}$.

Hence, show: i $\cos x = \dfrac{1}{2}(e^{ix}+e^{-ix})$ ii $\sin x = \dfrac{1}{2i}(e^{ix}-e^{-ix})$.

* questions that are intended for revision. They contain concepts not yet addressed.

Exercise A.9.2

1. Show the following complex numbers on an Argand diagram:

g $\frac{1}{2i}$ h $\frac{2}{1+i}$

3. If $z_1 = 1+2i$ and $z_2 = 1+i$, show each of the following on an Argand diagram:

g $\frac{z_1}{z_2}$ h $\frac{z_2}{z_1}$

4. Find the modulus and argument of:

d $3i$ e $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$ f $\frac{1}{\sqrt{2}}(i+1)$

g 6 h $\left(1-\frac{1}{2}i\right)^2$

14. Determine the modulus and argument of each of the complex numbers:

a $3-4i$ b $\frac{2}{1+i}$ c $\frac{1-i}{1+i}$

15. If $z = 1+i$ find $Arg(z)$. hence, find $Arg\left(\frac{1}{z^4}\right)$.

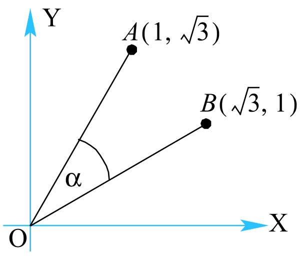
16. Determine the modulus and argument of each of the complex numbers:

a $\cos\theta + i\sin\theta$ b $\sin\theta + i\cos\theta$ c $\cos\theta - i\sin\theta$

17. Find the modulus and argument of:

a $1+i\tan\alpha$ b $\tan\alpha - i$ c $1+\cos\theta+i\sin\theta$

18. i Express $\frac{1+\sqrt{3}i}{\sqrt{3}+i}$ in the form $u+vi$.



ii Let α be the angle as shown in the diagram. Use part i to find α, clearly explaining your reason(s).

Hence, find $Arg(z)$ where $z = \left(\frac{1+\sqrt{3}i}{\sqrt{3}+i}\right)^7$.

19. Find:

i the modulus

ii the principal argument of the complex number $1 - \cos\theta - i\sin\theta$.

On an Argand diagram, for the case $0 < \theta < \pi$, interpret geometrically the relationship:

$$1 - \cos\theta - i\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\left(\cos\left(\frac{\theta - \pi}{2}\right) + i\sin\left(\frac{\theta - \pi}{2}\right)\right).$$

20. If $z = \cos\theta + i\sin\theta$, prove:

a $\dfrac{2}{1+z} = 1 - i\tan\left(\dfrac{\theta}{2}\right)$.

b $\dfrac{1+z}{1-z} = i\cot\left(\dfrac{\theta}{2}\right)$.

Exercise A.9.4

9. a If $z = cis(\theta)$, show that:

i $z^2 = \cos(2\theta) + i\sin(2\theta)$

ii $z^2 = (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta)$

Hence, show that:

A $\sin 2\theta = 2\sin\theta\cos\theta$ B $\cos 2\theta = \cos^2\theta - \sin^2\theta$ C $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

Using the same approach as that in part a, derive the following identities.

a $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

b $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

c $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

10. If $z = cis\theta$, prove: i $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ ii $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$

11. If $z = x + iy, y \neq 0$, show that $w = \dfrac{z}{(1 + z^2)}$, $1 + z^2 \neq 0$, is real, only if $|z| = 1$.

12. Simplify the expression $\dfrac{1 + i\tan\theta}{1 - i\tan\theta}$. Hence, show that $\left(\dfrac{1 + i\tan\theta}{1 - i\tan\theta}\right)^k = \dfrac{1 + i\tan k\theta}{1 - i\tan k\theta}$ where k is a positive integer.

13. Consider the complex number $z = cis\left(\dfrac{2k\pi}{5}\right)$ for any integer k such that $z \neq 1$.

a Show that $z^n + \dfrac{1}{z^n} = 2\cos\left(\dfrac{2nk\pi}{5}\right)$ for any integer n.

b Show that $z^5 = 1$. Hence, or otherwise, show that $1 + z + z^2 + z^3 + z^4 = 0$.

c Find the value of b, given that $\left(z + \dfrac{1}{z}\right)^2 + \left(z^2 + \dfrac{1}{z^2}\right)^2 = b$.

14. If n is a positive integer, show that $(1 + i)^n + (1 - i)^n = 2^{\frac{n}{2} + 1} \cdot \cos\left(\dfrac{n\pi}{4}\right)$.

15. Simplify the expression $\dfrac{(1 + cis(-\theta))^3}{(1 + cis(\theta))^3}$.

16. If $u = cis\theta$ and $v = cis\alpha$ express $\dfrac{u}{v} + \dfrac{v}{u}$ in terms of θ and α.

17. a If $cis\alpha = a$ and $cis\beta = b$, prove $\sin(\alpha - \beta) = \dfrac{b^2 - a^2}{2ab}i$.

b If $(1+cis\theta)(1+cis2\theta) = a+bi$, prove $a^2+b^2 = 16\cos^2\theta\cos^2\left(\frac{\theta}{2}\right)$.

18. If $|z| = 1$ and $Arg(z) = \theta, 0<\theta<\frac{\pi}{2}$, find: a $\left|\frac{2}{1-z^2}\right|$ b $\arg\left(\frac{2}{1-z^2}\right)$

Exercise A.9.5

8. The Property Index indicates the value of one unit invested in property. The Share index represents the value of one unit invested in the share market. A person has one unit of property and one of shares, the the value of the investment os the sum of the two values.

Years	Property Index
0.000	18.020
0.200	18.066
0.400	18.418
0.600	19.019
0.800	19.776
1.000	20.567
1.200	21.269
1.400	21.771
1.600	21.993
1.800	21.900
2.000	21.508
2.200	20.877
2.400	20.108
2.600	19.322
2.800	18.643
3.000	18.178
3.200	18.001

Years	Share Index
0.000	13.621
0.200	12.510
0.400	11.318
0.600	10.234
0.800	9.429
1.000	9.030
1.200	9.100
1.400	9.627
1.600	10.529
1.800	11.664
2.000	12.851
2.200	13.904
2.400	14.657
2.600	14.990
2.800	14.851
3.000	14.262
3.200	13.316

a Assume that the figures are periodic and can be modelled by functions of the form:

$I_{\text{Property}} = A_P\cos(nt+B_P)+C_P$ and $I_{\text{Shares}} = A_S\cos(nt+B_S)+C_S$.

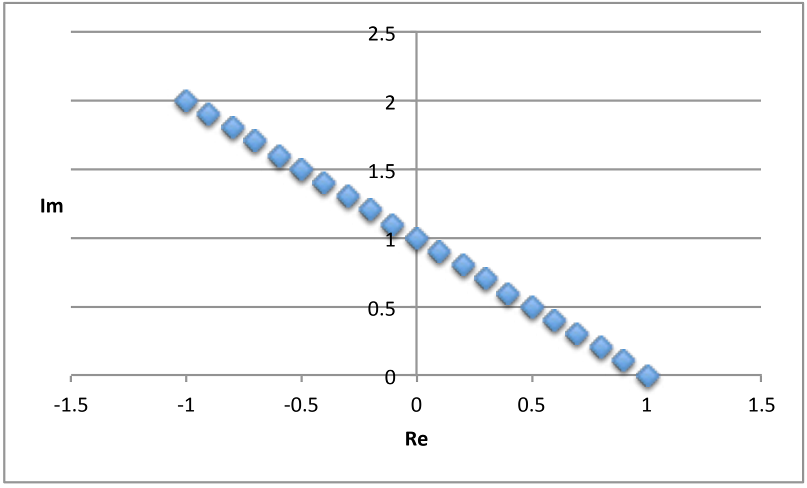
Find appropriate values for the parameters.

b Evaluate the appropriateness of your models.

c Find a function of the form: $I_{\text{Total}} = A_T\cos(nt+B_T)+C_T$ that models the total value of the portfolio.

d Evaluate the appropriateness of your final mode.

9. The following Argand Diagram shows a set of complex numbers.



Transform each of these numbers using the function $f(z)=z+\frac{1}{z}$. Show the transformed set on an Argand Diagram.

10. The number of sales per week (thousands) of Icecream and Chocolate by a Confectioner are:

Months	Icecream
0	21.000
1	20.471
2	19.025
3	17.043
4	15.050
5	13.572
6	13.001
7	13.487
8	14.901
9	16.870
10	18.874
11	20.382
12	20.996
13	20.554
14	19.172
15	17.216
16	15.203

Months	Chocolate
0	7.480
1	7.015
2	7.869
3	9.814
4	12.338
5	14.772
6	16.473
7	16.992
8	16.191
9	14.282
10	11.770
11	9.319
12	7.576
13	7.003
14	7.751
15	9.622
16	12.122

a Assume that the figures are periodic and can be modelled by functions of the form:

$S_{\text{Icecream}} = A_{\text{I}}\cos\left(nt + B_{I}\right) + C_{\text{I}}$ and $S_{\text{Chocolate}} = A_{\text{C}}\cos\left(nt + B_{\text{C}}\right) + C_{\text{C}}$.

Find appropriate values for the parameters.

b Evaluate the appropriateness of your models.

c Find a function of the form: $S_{\text{Total}} = A_{\text{T}}\cos\left(nt + B_{\text{T}}\right) + C_{\text{T}}$ that models the total value of the portfolio.

d Evaluate the appropriateness of your final mode.

Exercise B.6.1

3 All of the following functions are mappings of $\mathbb{R}\mapsto\mathbb{R}$ unless otherwise stated.

a Determine the composite functions $(f\circ g)(x)$ and $(g\circ f)(x)$, if they exist.

b For the composite functions in part a that do exist, find their range.

viii $f(x) = x-4,\ g(x) = |x|$

ix $f(x) = x^3-2,\ g(x) = |x+2|$

x $f(x) = \sqrt{4-x},\ x\le 4,\ g(x) = x^2$

xi $f(x) = \dfrac{x}{x+1},\ x\ne -1,\ g(x) = x^2$

xii $f(x) = x^2+x+1,\ g(x) = |x|$

xiii $f(x) = 2^x,\ g(x) = x^2$

xiv $f(x) = \dfrac{1}{x+1},\ x\ne -1,\ g(x) = x-1$

xv $f(x) = \dfrac{2}{\sqrt{x-1}},\ x>1,\ g(x) = x^2+1$

xvi $f(x) = 4^x,\ g(x) = \sqrt{x}$

13 Find $(h\circ f)(x)$, given that $h(x) = \begin{cases} x^2+4, & x\ge 1 \\ 4-x, & x<1 \end{cases}$ and $f: x\mapsto x-1,\ x\in\mathbb{R}$.

Sketch the graph of $(h\circ f)(x)$ and use it to find its range.

14 a Given three functions, f, g and h, when would $h\circ g\circ f$ exist?

b If $f: x\mapsto x+1,\ x\in\mathbb{R}$, $g: x\mapsto x^2,\ x\in\mathbb{R}$ and $h: x\mapsto 4x,\ x\in\mathbb{R}$, find $(h\circ g\circ f)(x)$.

15 Given the functions $f(x) = e^{2x-1}$ and $g(x) = \frac{1}{2}(\ln x+1)$ find, where they exist:

a $(f\circ g)$ b $(g\circ f)$ c $(f\circ f)$

In each case find the range of the composite function.

16 Given that $h(x) = \log_{10}(4x-1),\ x>\frac{1}{4}$ and $k(x) = 4x-1,\ x\in\ ]-\infty,\infty[$, find, where they exist :

a $(h\circ k)$ b $(k\circ h)$.

17 Given the functions $f(x) = \sqrt{x^2-9},\ x\in\mathbf{S}$ and $g(x) = |x|-3,\ x\in\mathbf{T}$, find the largest positive subsets of $\mathbb{R}$ so that:

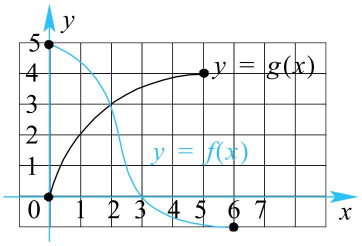
a $g\circ f$ exists b $f\circ g$ exists.

18 For each of the following functions:

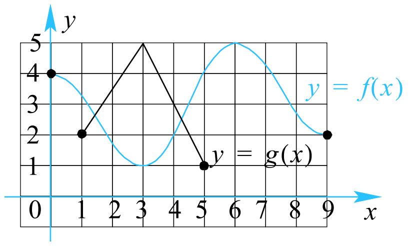
a determine if $f\circ g$ exists and sketch the graph of $f\circ g$ when it exists.

b determine if $g\circ f$ exists and sketch the graph of $g\circ f$ when it exists.

i



ii



19 Given the functions $f : S \to \mathbb{R}$ where $f(x) = e^{x+1}$ and $g : S \to \mathbb{R}$ where $g(x) = \ln 2x$ where $S =]0, \infty[$.

a State the domain and range of both f and g.

b Giving reasons, show that $g \circ f$ exists but $f \circ g$ does not exist.

c Fully define $g \circ f$, sketch its graph and state its range.

20 The functions *f* and *g* are given by $f(x) = \begin{cases} \sqrt{x-1} & \text{if } x \geq 1 \\ x-1 & \text{if } 0 < x < 1 \end{cases}$ and $g(x) = x^2 + 1$.

a Show that $f \circ g$ is defined. b Find $(f \circ g)(x)$ and determine its range.

21 Let $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ where $f(x) = \begin{cases} \dfrac{1}{x^2}, & 0 < x \leq 1 \\ \dfrac{1}{\sqrt{x}}, & x > 1 \end{cases}$.

a Sketch the graph of *f*.

b Define the composition $f \circ f$, justifying its existence.

c Sketch the graph of $f \circ f$, giving its range.

22 Consider the functions $f :]1, \infty[\to \mathbb{R}$ where $f(x) = \sqrt{x}$ and $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ where $g(x) = x^2$.

a Sketch the graphs of *f* and *g* on the same set of axes.

b Prove that $g \circ f$ exists and find its rule.

c Prove that $f \circ g$ cannot exist.

d If a new function $g^* : S \to \mathbb{R}$ where $g^*(x) = g(x)$ is now defined, find the largest positive subset of $\mathbb{R}$ so that $f \circ g^*$ does exist. Find $f \circ g^*$, sketch its graph and determine its range.

23 Given that $f(x) = \dfrac{ax - b}{cx - a}$, show that $f \circ f$ exists and find its rule.

24 a Sketch the graphs of $f(x) = \dfrac{1}{a}x^2$ and $g(x) = \sqrt{2a^2 - x^2}$, where $a > 0$.

b Show that $f \circ g$ exists, find its rule and state its domain.

c Let S be the largest subset of $\mathbb{R}$ so that $g \circ f$ exists.

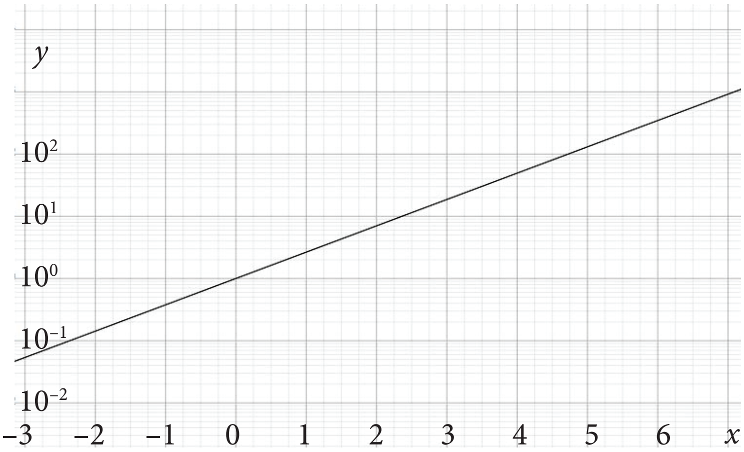
i Find S.

ii Fully define $g \circ f$, sketch its graph and find its range.

Exercise B.8.3

For these questions, find appropriate models for the data.

8.



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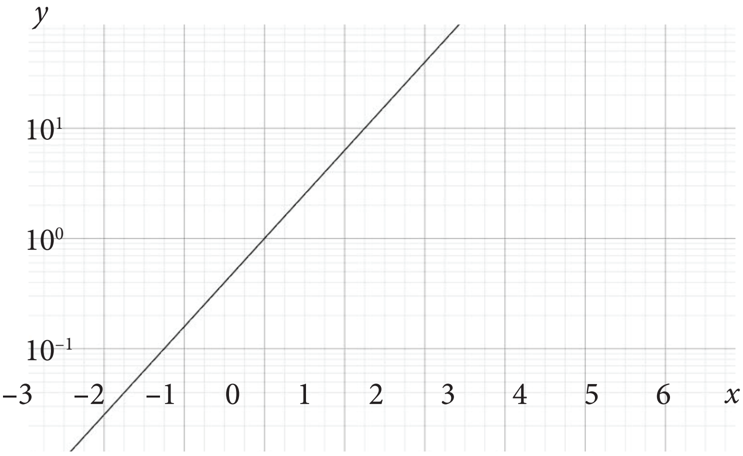

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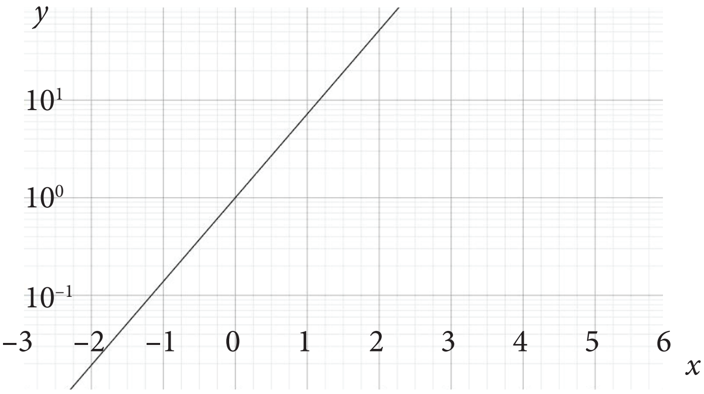

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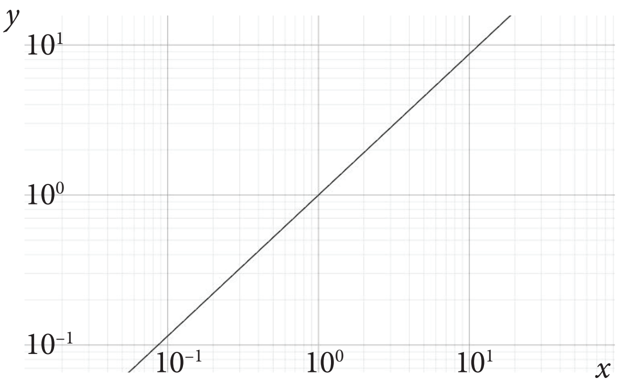

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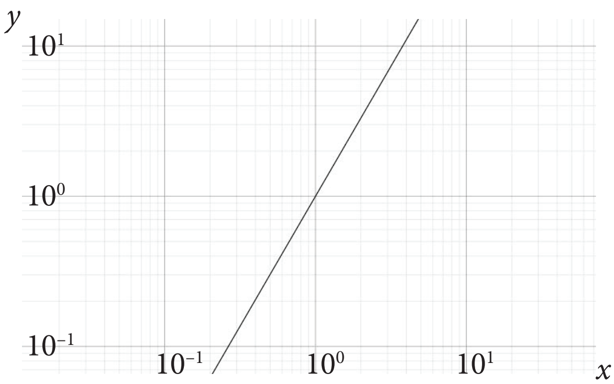

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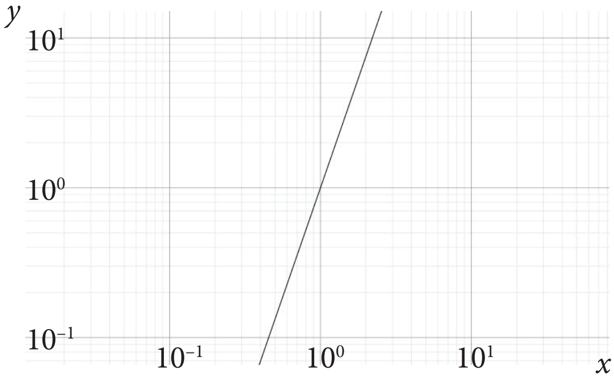


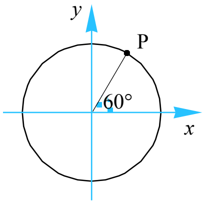
14. ` ` `

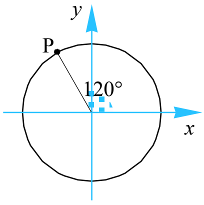
x	y
1.3	3.2397
1.4	3.5463
1.5	3.8819
1.6	4.2493
1.7	4.6514
1.8	5.0916
1.9	5.5735
2	6.1009
2.1	6.6783
2.2	7.3103
2.3	8.0021
2.4	8.7594
2.5	9.5883

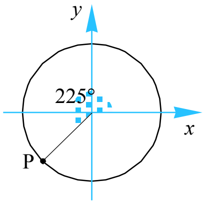
Exercise C.8.1

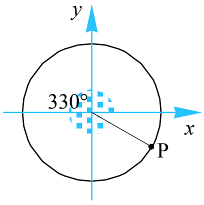
7. Find the coordinates of the point P on the following unit circles.

a **b** **c** **d**









8. Find the exact value of:

a $\sin\frac{11\pi}{6}\cos\frac{5\pi}{6} - \sin\frac{5\pi}{6}\cos\frac{11\pi}{6}$ **b** $2\sin\frac{\pi}{6}\cos\frac{\pi}{6}$

c $\dfrac{\tan\frac{\pi}{3} - \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{3}\tan\frac{\pi}{6}}$ **d** $\cos\frac{\pi}{4}\cos\frac{\pi}{3} + \sin\frac{\pi}{4}\sin\frac{\pi}{3}$

9. Show that the following relationships are true.

a $\sin 2\theta = 2\sin\theta\cos\theta$, where $\theta = \frac{\pi}{3}$ **b** $\cos 2\theta = 2\cos^2\theta - 1$, where $\theta = \frac{\pi}{6}$

c $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$, where $\theta = \frac{2\pi}{3}$ **d** $\sin(\theta - \phi) = \sin\theta\cos\phi - \sin\phi\cos\theta$, where $\theta = \frac{2\pi}{3}$ and $\phi = -\frac{\pi}{3}$

10. Given that $\sin\theta = \frac{2}{3}$ and $0 < \theta < \frac{\pi}{2}$, find:

a $\sin(\pi + \theta)$ **b** $\sin(2\pi - \theta)$ **c** $\cos\left(\frac{\pi}{2} + \theta\right)$

11. Given that $\cos\theta = \frac{2}{5}$ and $0 < \theta < \frac{\pi}{2}$, find:

a $\cos(\pi - \theta)$ **b** $\sec\theta$ **c** $\sin\left(\frac{\pi}{2} - \theta\right)$

12. Given that $\tan\theta = k$ and $0 < \theta < \frac{\pi}{2}$, find:

a $\tan(\pi + \theta)$ **b** $\tan\left(\frac{\pi}{2} + \theta\right)$ **c** $\tan(-\theta)$

13. Given that $\sin\theta = \frac{2}{3}$ and $0 < \theta < \frac{\pi}{2}$, find:

a $\cos\theta$ **b** $\sec\theta$ **c** $\cos(\pi + \theta)$

14. Given that $\cos\theta = -\frac{4}{5}$ and $\pi < \theta < \frac{3\pi}{2}$, find:

a $\sin\theta$ **b** $\tan\theta$ **c** $\cos(\pi + \theta)$

15. Given that $\tan\theta = -\frac{4}{3}$ and $\frac{\pi}{2} < \theta < \pi$, find:

a $\sin\theta$ **b** $\tan\left(\frac{\pi}{2} + \theta\right)$ **c** $\sec\theta$

16. Given that $\cos\theta = k$ and $\frac{3\pi}{2} < \theta < 2\pi$, find:

a $\cos(\pi - \theta)$ **b** $\sin\theta$ **c** $\cot\theta$

17. Given that $\sin\theta = -k$ and $\pi < \theta < \frac{3\pi}{2}$,

find:

a $\cos\theta$ **b** $\tan\theta$ **c** $\operatorname{cosec}\left(\frac{\pi}{2} + \theta\right)$

18. Simplify the following.

a $\dfrac{\sin(\pi - \theta)\cos\left(\frac{\pi}{2} + \theta\right)}{\sin(\pi + \theta)}$ **b** $\dfrac{\sin\left(\frac{\pi}{2} + \theta\right)\cos\left(\frac{\pi}{2} - \theta\right)}{\sin^2\theta}$ **c** $\dfrac{\sin\left(\frac{\pi}{2} - \theta\right)}{\cos\theta}$

d $\tan(\pi + \theta)\cot\theta$ **e** $\cos(2\pi - \theta)\operatorname{cosec}\theta$ **f** $\dfrac{\sec\theta}{\operatorname{cosec}\theta}$

19. If $0 \le \theta \le 2\pi$, find all values of x such that:

a $\sin x = \frac{\sqrt{3}}{2}$ **b** $\cos x = \frac{1}{2}$ **c** $\tan x = \sqrt{3}$

d $\cos x = -\frac{\sqrt{3}}{2}$ **e** $\tan x = -\frac{1}{\sqrt{3}}$ **f** $\sin x = -\frac{1}{2}$

Exercise C.8.2

6. Prove $\sin^2x(1+n\cot^2x)+\cos^2x(1+n\tan^2x) = \sin^2x(n+\cot^2x)+\cos^2x(n+\tan^2x)$.

7. If $k\sec\phi = m\tan\phi$, prove that $\sec\phi\tan\phi = \frac{mk}{m^2-k^2}$.

8. If $x = k\sec^2\phi + m\tan^2\phi$ and $y = l\sec^2\phi + n\tan^2\phi$, prove that $\frac{x-k}{k+m} = \frac{y-l}{l+n}$.

9. Given that $\tan\theta = \frac{2a}{a^2-1}, 0<\theta<\frac{\pi}{2}$, find: **a** $\sin\theta$ **b** $\cos\theta$

10. **a** If $\sin x + \cos x = 1$, find the values of: **i** $\sin^3x+\cos^3x$ **ii** $\sin^4x+\cos^4x$

b Hence, deduce the value of $\sin^kx+\cos^kx$, where k is a positive integer.

11. If $\tan\phi = -\frac{1}{\sqrt{x^2-1}}, \frac{\pi}{2}<\phi<\pi$, find, in terms of *x*,

a $\sin\phi+\cos\phi$ b $\sin\phi-\cos\phi$ c $\sin^4\phi-\cos^4\phi$

12. Find: **a** the maximum value of **b** the minimum value of

i $\cos^2\theta+5$ **ii** $\frac{5}{3\sin^2\theta+2}$ **iii** $2\cos^2\theta+\sin\theta-1$

13. **a** Given that $b\sin\phi = 1$ and $b\cos\phi = \sqrt{3}$, find *b*.

b Hence, find all values of ϕ that satisfy the relationship described in part **a**.

14. Find: **a** the maximum value of **b** the minimum value of

i $5^{3\sin\theta-1}$ **ii** $3^{1-2\cos\theta}$

15. Given that $\sin\theta\cos\theta = k$, find: **a** $(\sin\theta+\cos\theta)^2$, $\sin\theta+\cos\theta>0$.

b $\sin^3\theta+\cos^3\theta$, $\sin\theta+\cos\theta>0$ b

16. **a** Given that $\sin\phi = \frac{1-a}{1+a}$, $0 < \phi < \frac{\pi}{2}$, find $\tan\phi$.

    b Given that $\sin\phi = 1 - a$, $\frac{\pi}{2} < \phi < \pi$, find : **i** $2 - \cos\phi$ **ii** $\cot\phi$

17. Find:

    a the value(s) of $\cos x$, where $\cot x = 4(\operatorname{cosec} x - \tan x)$, $0 < x < \pi$.

    b the values of $\sin x$, where $3\cos x = 2 + \frac{1}{\cos x}$, $0 \le x \le 2\pi$.

18. Given that $\sin 2x = 2\sin x\cos x$, find all values of x, such that $2\sin 2x = \tan x$, $0 \le x \le \pi$.

Exercise C.9.6

11. The line L is defined by the parametric equations $x = 4-5k$ and $y = -2+3k$.

a Find the coordinates of three points on L.

b Find the value of k that corresponds to the point (14, –8).

c Show that the point (–1, 4) does not lie on the line L.

d Find the vector form of the line L.

e A second line, M, is defined parametrically by $x = a+10\lambda$ and $y = b-6\lambda$. Describe the relationship between M and L for the case that:

i $a = 8$ and $b = 4$ ii $a = 4$ and $b = -2$

12. Find the Cartesian equation of the line that passes through the point A(2, 1) and such that it is perpendicular to the vector $4\boldsymbol{i}+3\boldsymbol{j}$.

13. Find the direction cosines for each of the following lines:

a $\boldsymbol{r} = \begin{pmatrix}3\\4\end{pmatrix}+\mu\begin{pmatrix}-3\\2\end{pmatrix}$ **b** $\boldsymbol{r} = \begin{pmatrix}5\\9\end{pmatrix}+\lambda\begin{pmatrix}4\\3\end{pmatrix}$.

14. a Show that the line $ax+by+c = 0$ has a directional vector $\begin{pmatrix}b\\-a\end{pmatrix}$ and a normal vector $\begin{pmatrix}a\\b\end{pmatrix}$.

b By making use of directional vectors, which of the following lines are parallel to $L:2x+3y = 10$?

i $5x-2y = 10$

ii $6x+9y = 20$

iii $4x+6y = -10$

15. Find the point of intersection of the lines $\boldsymbol{r} = \begin{pmatrix}-2\\1\end{pmatrix}+\lambda\begin{pmatrix}3\\8\end{pmatrix}$ and $\frac{x-3}{2} = \frac{y}{5}$.

16. Find a vector equation of the line passing through the origin that also passes through the point of intersection of the lines:

$$\boldsymbol{u} = \begin{pmatrix}3\\2\end{pmatrix}+\lambda\begin{pmatrix}-1\\3\end{pmatrix} \text{ and } \boldsymbol{v} = \begin{pmatrix}-1\\1\end{pmatrix}+\mu\begin{pmatrix}2\\1\end{pmatrix} .$$

17. Consider the line with vector equation $\boldsymbol{r} = (4\boldsymbol{i}-3\boldsymbol{j})+\lambda(3\boldsymbol{i}+4\boldsymbol{j})$. Find the points of intersection of this line with the line:

a $\boldsymbol{u} = (4\boldsymbol{i}+5\boldsymbol{j})+\mu(2\boldsymbol{i}-\boldsymbol{j})$

b $\quad \boldsymbol{v} = (-2\boldsymbol{i} + 3\boldsymbol{j}) + t(-6\boldsymbol{i} - 8\boldsymbol{j})$

c $\quad \boldsymbol{w} = (13\boldsymbol{i} + 9\boldsymbol{j}) + s(3\boldsymbol{i} + 4\boldsymbol{j})$

Exercise C.9.7

8. Show that the lines $\frac{x-1}{2}=2-y=5-z$ and $\frac{4-x}{4}=\frac{3+y}{2}=\frac{5+z}{2}$ are parallel.

9. Find the Cartesian equation of the lines joining the points

a (−1, 3, 5) to (1, 4, 4) b (2, 1, 1) to (4, 1, −1)

10. a Find the coordinates of the point where the line $\boldsymbol{r}=\begin{pmatrix}-2\\5\\3\end{pmatrix}+t\begin{pmatrix}-1\\2\\1\end{pmatrix}$ intersects the x-y plane.

b The line $\frac{x-3}{4}=y+2=\frac{4-z}{5}$ passes through the point $(a, 1, b)$. Find the values of a and b.

11. Find the Cartesian equation of the line having the vector form:

a $\boldsymbol{r}=\begin{pmatrix}1\\4\\-2\end{pmatrix}+t\begin{pmatrix}1\\-1\\0\end{pmatrix}$ b $\boldsymbol{r}=\begin{pmatrix}2\\1\\3\end{pmatrix}+t\begin{pmatrix}2\\0\\0\end{pmatrix}$.

In each case, provide a diagram showing the lines.

12. Find the vector equation of the line represented by the Cartesian form $\frac{x-1}{2}=\frac{1-2y}{3}=z-2$.

Clearly describe this line.

13. Find the acute angle between the following lines.

a $\boldsymbol{r}=\begin{pmatrix}0\\2\\3\end{pmatrix}+s\begin{pmatrix}3\\4\\5\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}-2\\5\\3\end{pmatrix}+t\begin{pmatrix}-1\\2\\1\end{pmatrix}$.

b $\boldsymbol{r}=\begin{pmatrix}2\\1\\4\end{pmatrix}+s\begin{pmatrix}-2\\0\\1\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}1\\1\\1\end{pmatrix}+s\begin{pmatrix}1\\1\\3\end{pmatrix}$

c $\frac{x-3}{-1}=\frac{2-y}{3}=\frac{z-4}{2}$ and $\frac{x-1}{2}=\frac{y-2}{-2}=z-2$

14. Find the point of intersection of the lines:

a $\frac{x-5}{-2} = y-10 = \frac{z-9}{12}$ and $x = 4, \frac{y-9}{-2} = \frac{z+9}{6}$

b $\frac{2x-1}{3} = \frac{y+5}{3} = \frac{z-1}{-2}$ and $\frac{2-x}{4} = \frac{y+3}{2} = \frac{4-2z}{1}$

15. Find the Cartesian form of the lines with parametric equation given by:

L : $x = \lambda, y = 2\lambda+2, z = 5\lambda$ and M : $x = 2\mu-1, y = -1+3\mu, z = 1-2\mu$

a Find the point of intersection of these two lines.

b Find the acute angle between these two lines.

c Find the coordinates of the point where: i L cuts the x-y plane. ii M cuts the y-z plane.

16. Show that the lines $\frac{x-2}{3} = \frac{y-3}{-2} = \frac{z+1}{5}$ and $\frac{x-5}{-3} = \frac{y-1}{2} = \frac{z-4}{-5}$ are coincident.

17. Show that the lines $\frac{x-1}{-3} = y-2 = \frac{7-z}{11}$ and $\frac{x-2}{3} = \frac{y+1}{8} = \frac{z-4}{-7}$ are skew.

18. Find the equation of the line passing through the origin and the point of intersection of the lines with equations

$x-2 = \frac{y-1}{4}, z = 3$ and $\frac{x-6}{2} = y-10 = z-4$.

19. The lines $\frac{x}{3} = \frac{y-2}{4} = 3+z$ and $x = y = \frac{z-1}{2k}$, $k \in \mathbb{R}\backslash\{0\}$ meet at right angles. Find k.

20. Consider the lines L : $x = 0, \frac{y-3}{2} = z+1$ and M : $\frac{x}{4} = \frac{y}{3} = \frac{z-10}{-1}$.

Find, correct to the nearest degree, the angle between the lines L and M.

21. Find the value(s) of k, such that the lines $\frac{x-2}{k} = \frac{y}{2} = \frac{3-z}{3}$ and $\frac{x}{k-1} = \frac{y+2}{3} = \frac{z}{4}$ are perpendicular.

22. Find a direction vector of the line that is perpendicular to both $\frac{x+1}{3} = \frac{y+1}{8} = \frac{z+1}{12}$ and $\frac{1-2x}{-4} = \frac{3y+1}{9} = \frac{z}{6}$.

23. Are the lines $\frac{x-1}{5} = \frac{y+2}{4} = \frac{4-z}{3}$ and $\frac{x+2}{3} = \frac{y+7}{2} = \frac{2-z}{3}$ parallel? Find the point of intersection of these lines.

What do you conclude?

Exercise D.12.1

13. Faults occur randomly along the length of a yarn of wool where the number of faults per bobbin holding a fixed length of yarn may be assumed to follow a Poisson distribution. A bobbin is rejected if it contains at least one fault. It is known that in the long run 33% of bobbins are rejected.

    a Find the probability that a rejected bobbin contains only one fault.

    The production manager believes that by doubling the length of yarn on each bobbin there will be a smaller rejection rate. Assuming that the manufacturing process has not altered, is the production manager correct?

    Provide a quantitative argument.

14. On average, it is found that 8 out of every 10 electric components produced from a large batch have at least one defective component. Find the probability that there will be at least 2 defective components from a randomly selected batch.

15. Flaws, called seeds, in a particular type of glass sheet occur at a rate of 0.05 per square metre. Find the probability that a rectangular glass sheet measuring 4 metres by 5 metres contains:

    a no seeds.

    b at least two seeds.

    Sheets containing at least two seeds are rejected.

    c Find the probability that, in a batch of ten such glass sheets, at most one is rejected.

16. Simar has decided to set up a small business venture. The venture requires Simar to go fishing every Sunday so he can sell his catch on the Monday. He realises that on a proportion p of these days he does not catch anything.

    a Find the probability that on any given Sunday, Simar catches:

    i no fish. ii one fish. iii at least two fish.

    The cost to Simar on any given Sunday if he catches no fish is \$5. If he catches one fish Simar makes a profit of \$2 and if he catches more than one fish he makes a profit of \$10. Let the random variable X denote the profit Simar makes on any given Sunday.

    b Show that $E(X) = 10 - 15p + 8p\ln p, 0 < p < 1$

    c Find the maximum value of p, if Simar is to make a positive gain on his venture.

Exercise E.10.3

3. Differentiate the following.

g $\frac{4}{x^2}\times\sin x$ h $xe^x\sin x$ i $xe^x\log_e x$

4. Differentiate the following.

g $\frac{e^x-1}{x+1}$ h $\frac{\sin x+\cos x}{\sin x-\cos x}$ i $\frac{x^2}{x+\log_e x}$

5. Differentiate the following.

f $\cos(-4x)-e^{-3x}$ g $\log_e(4x+1)-x$ h $\log_e(e^{-x})+x$

i $\sin\left(\frac{x}{2}\right)+\cos(2x)$ j $\sin(7x-2)$ k $\sqrt{x}-\log_e(9x)$

l $\log_e(5x)-\cos(6x)$

6. Differentiate the following.

i $\cos(\sin\theta)$ j $4\sec\theta$ k $\operatorname{cosec}(5x)$ l $3\cot(2x)$

7. Differentiate the following.

k $e^{-\cos(2\theta)}$ l $e^{2\log_e(x)}$ m $\frac{2}{e^{-x}+1}$

n $(e^x-e^{-x})^3$ o $\sqrt{e^{2x+4}}$ p e^{-x^2+9x-2}

8. Differentiate the following.

i $\log_e\left(\frac{1}{\sqrt{x+2}}\right)$ j $\log_e(\cos^2 x+1)$ k $\log_e(x\sin x)$ l $\log_e\left(\frac{x}{\cos x}\right)$

9. Differentiate the following.

i $\frac{\cos(2x)}{e^{1-x}}$ j $x^2\log_e(\sin 4x)$ k $e^{-\sqrt{x}}\sin\sqrt{x}$ l $\cos(2x\sin x)$

m $\frac{e^{5x+2}}{1-4x}$ n $\frac{\log_e(\sin\theta)}{\cos\theta}$ o $\frac{x}{\sqrt{x+1}}$ p $x\sqrt{x^2+2}$

q $(x^3+x)\sqrt[3]{x+1}$ r $(x^3-1)\sqrt{x^3+1}$ s $\frac{1}{x}\log_e(x^2+1)$

t $\log_e\left(\frac{x^2}{x^2+2x}\right)$ u $\frac{\sqrt{x-1}}{x}$ v $e^{-x}\sqrt{x^2+9}$

w $(8 - x^3)\sqrt{2 - x}$ x $x^n \ln(x^n - 1)$

15. Find: c $\frac{d}{dx}(\cos x°)$

17. a Given that $f(x) = 1 - x^3$ and $g(x) = \log_e x$, find: i $(f\circ g)'(x)$ ii $(g\circ f)'(x)$

b Given that $f(x) = \sin(x^2)$ and $g(x) = e^{-x}$, find: i $(f\circ g)'(x)$ ii $(g\circ f)'(x)$

21. Differentiate the following.

e $y = \cot\left(\frac{\pi}{4} - x\right)$ f $y = \sec(2x - \pi)$

22. Differentiate the following.

g $x^4 \operatorname{cosec}(4x)$ h $\tan 2x \cot x$ i $\sqrt{\sec x + \cos x}$

23. Differentiate the following.

a $e^{\sec x}$ b $\sec(e^x)$ c $e^x \sec x$

d $\cot(\ln x)$ e $\ln(\cot 5x)$ f $\cot x \ln x$

g $\operatorname{cosec}(\sin x)$ h $\sin(\operatorname{cosec} x)$ i $\sin x \operatorname{cosec} x$

Exercise E.10.4

17. A solid ball of radius 30 cm is dissolving uniformly in such a way that its radius is x cm, and is decreasing at a constant rate of 0.15 cm/s, t seconds after the process started.

  a Find an expression for the radius of the ball at any time t seconds.

  b Find the domain of x.

  c Find the rate of change of:

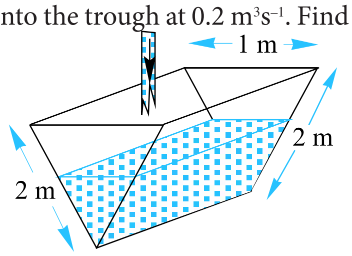
    i the volume of the ball 10 seconds after it started to dissolve.

    ii the surface area when the ball has a volume of 100π cm^3.

  d Sketch a graph of the volume of the ball at time t seconds.

18. A fisherman is standing on a jetty and is pulling in a boat by means of a rope passing over a pulley. The pulley is 3 m above the horizontal line where the rope is tied to the boat. At what rate is the boat approaching the jetty if the rope is being hauled at 1.2 m/s, when the rope measures 12 m?

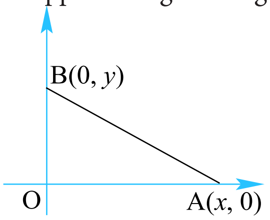
19. A trough, 4 m long, has a cross-section in the shape of an isosceles triangle. Water runs into the trough at 0.2 m^{3}s^{-1}. Find the rate at which the water level is rising after 10 seconds if the tank is initially empty.



20. A line, 12 m long, meets the x-axis at A and the y-axis at B. If point A, initially 5 m from O, is approaching the origin, O, at 2 m/s, find:

  a an expression for y in terms of the time, t seconds, since point A started to move.

  b the rate at which B is moving when A has travelled 2 m.



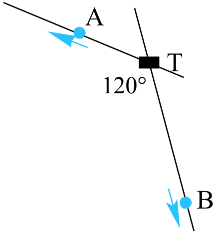
21^: The volume V cm^3 of water in a container at time t seconds, when the depth of water in the container is x cm is given by the relationship

$$V = \frac{1}{3}(x+3)^3 - 9, 0 \leq x \leq 5.$$

  a Find the rate at which the water level is increasing after 5 seconds if water flows into the container at 1.2 cm^{3}s^{-1}.

  b Find the rate of change of the area of the surface of the water after 5 seconds if water is still flowing into the container at 1.2 cm^{3}s^{-1}.

22. Two cars, A and B, leave their hometown, T, at the same time but on different freeways. The freeways are straight and at 120° to each other and the cars are travelling at 70 km/h and 80 km/hr respectively. Given that x km and y km are the distances travelled by the cars A and B respectively t hours after they leave T:



  a find an expression in terms of t for the distance travelled by car:

    i A ii B

  b find an expression in terms of t for the distance apart cars A and B are after t hours.

  c How fast are cars A and B moving apart after 5 hours?

  d After travelling for 5 hours, the driver of car B decides to head back to T. How fast are the cars moving apart 3 hours after car B turns back?

23. A girl approaches a tower 75 m high at 5 km/hr. At what rate is her distance from the top of the tower changing when she is 50 m from the foot of the tower?

24. Jenny is reeling in her kite, which is maintaining a steady height of 35 m above the reel. If the kite has a horizontal speed of 0.8 m/s towards Jenny, at what rate is the string being reeled in when the kite is 20 m horizontally from Jenny?

25. A kite 60 metres high, is being carried horizontally away by a wind gust at a rate of 4 m/s. How fast is the string being let out when the string is 100 m long?

26. Grain is being released from a chute at the rate of 0.1 cubic metres per minute and is forming a heap on a level horizontal floor in the form of a circular cone that maintains a constant semi-vertical angle of 30°. Find the rate at which the level of the grain is increasing 5 minutes after the chute is opened.

27. A radar tracking station is located at ground level vertically below the path of an approaching aircraft flying at 850 km/h and maintaining a constant height of 9,000 m. At what rate in degrees is the radar rotating while tracking the plane when the horizontal distance of the plane is 4 km from the station.

28. A weather balloon is released at ground level and 2,500 m from an observer on the ground. The balloon rises straight upwards at 5 m/s. If the observer is tracking the balloon from his fixed position, find the rate at which the observer's tracking device must rotate so that it can remain in-line with the balloon when the balloon is 400 m above ground level.

29. The radius of a uniform spherical balloon is increasing at 3% per second.

    a Find the percentage rate at which its volume is increasing.

    b Find the percentage rate at which its surface area is increasing.

30. A manufacturer has agreed to produce x thousand 10-packs of high quality recordable compact discs and have them available for consumers every week with a wholesale price of $k per 10-pack. The relationship between x and k has been modelled by the equation $x^2 - 2.5kx + k^2 = 4.8$

    At what rate is the supply of the recordable compact discs changing when the price per 10-pack is set at $9.50, 4420 of the 10-pack discs are being supplied and the wholesale price per 10-pack is increasing at 12 cents per 10-pack per week?

31. It has been estimated that the number of housing starts, N millions, per year over the next 5 years will be given by:

$$N(r) = \frac{8}{1 + 0.03r^2},$$

    where r% is the mortgage rate. The government believes that over the next t months, the mortgage rate will be given by:

$$r(t) = \frac{8.6t + 65}{t + 10}.$$

    Find the rate at which the number of housing starts will be changing 2 years from when the model was proposed.

32. The volume of a right circular cone is kept constant while the radius of the base of the cone is decreasing at 2% per second. Find the percentage rate at which the height of the cone is changing.

33. The radius of a sector of fixed area is increasing at 0.5 m/s. Find the rate at which the angle in radians of the sector is changing when the ratio of the radius to the angle is 4.

Exercise E.11.1

1. Find the second derivative of the following functions.

m $f(x) = x^3 \sin x$ n $y = x\ln x$ o $f(x) = \dfrac{x^2-1}{2x+3}$ p $y = x^3 e^{2x}$

q $f(x) = \dfrac{\cos(4x)}{e^x}$ r $y = \sin(x^2)$ s $f(x) = \dfrac{x}{1-4x^3}$ t $y = \dfrac{x^2-4}{x-3}$

7. Find the *n*th derivative of:

a e^{ax} b $y = \dfrac{1}{2x+1}$ c $\sin(ax+b)$

8. a Find $f''(2)$ if $f(x) = x^2 - \sqrt{x}$. b Find $f''(1)$ if $f(x) = x^2\mathrm{Tan}^{-1}(x)$.

9. Find the rate of change of the gradient of the function $g(x) = \dfrac{x^2-1}{x^2+1}$ where $x = 1$.

10. Find the values of x where the rate of change of the gradient of the curve $y = x\sin x$ for $0 \le x \le 2\pi$ is positive.

Exercise E.11.2

2. Find the coordinates and nature of the stationary points for the following:

j $y = x\sqrt{x} - x, x \geq 0$ k $g(x) = x + \frac{4}{x}, x \neq 0$ l $f(x) = x^2 + \frac{1}{x^2}, x \neq 0$

3. Sketch the following functions:

e $f(x) = \frac{1}{3}x^3 - x^2 + 4$ f $y = 4x^3 - x^4$ g $y = x^3 - 8$

h $y = x^4 - 16$ i $y = x - 4x\sqrt{x}, x \geq 0$ j $f(x) = x - 2\sqrt{x}, x \geq 0$

4. Find m and n so that $f'(1)$ exists for the function $f(x) = \begin{cases} mx^2 + n & \text{if } x \leq 1 \\ \frac{1}{x} & \text{if } x > 1 \end{cases}$.

Exercise E.12.2

9. The acceleration, in m/s² of a body in a medium is given by $\frac{dv}{dt} = \frac{3}{t+1}, t \geq 0$. The particle has an initial speed of 6 m/s, find the speed (to 2 d.p) after 10 seconds.

10. The rate of change of the water level in an empty container, t seconds after it started to be filled from a tap is given by the relation:

$$\frac{dh}{dt} = 0.2\sqrt[3]{t+8}, t \geq 0$$

,

where h cm is the water level. Find the water level after 6 seconds.

11. The gradient function of the curve $y = f(x)$ is given by $e^{0.5x} - \cos(2x)$. Find the equation of the function, given that it passes through the origin.

12. a Given that $\frac{d}{dx}(e^{ax}(p\sin bx + q\cos bx)) = e^{ax}\sin bx$, express p and q in terms of a and b.

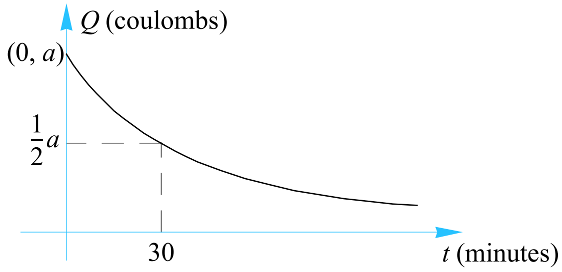
b Hence find $\int e^{2x}\sin 3x dx$.

13 The rate of change of the charge, Q, in coulombs, retained by a capacitor t minutes after charging, is given by $\frac{dQ}{dt} = -ake^{-kt}$.

Using the graph shown, determine the charge remaining after

a one hour

b 80 minutes



14. a Show that $\frac{d}{dx}(x\ln(x+k)) = \frac{x}{x+k} + \ln(x+k)$, where k is a real number.

b For a particular type of commercial fish, it is thought that a length–weight relationship exists such that their rate of change of weight, w kg, with respect to their length, x m, is modelled by the equation:

$$\frac{dw}{dx} = 0.2\ln(x+2).$$

Given that a fish in this group averages a weight of 650 gm when it is 20 cm long, find the weight of a fish measuring 30 cm.

15. The rate of flow of water, $\frac{dV}{dt}$ litres/hour, pumped into a hot water system over a 24-hour period from 6:00 am, is modelled by the relation:

$$\frac{dV}{dt} = 12 + \frac{3}{2}\cos\frac{\pi}{3}t\,,\ t \geq 0.$$

a Sketch the graph of $\frac{dV}{dt}$ against t.

b For what percentage of the time will the rate of flow exceed 11 litres/hour.

c How much water has been pumped into the hot water system by 8:00 a.m.?

16. The rates of change of the population size of two types of insect pests over a 4-day cycle, where t is measured in days, has been modelled by the equations:

$$\frac{dA}{dt} = 2\pi\cos\pi t,\ t \geq 0 \text{ and } \frac{dB}{dt} = \frac{3}{4}e^{0.25t},\ t \geq 0$$

where A and B represent the number of each type of pest in thousands.

Initially there were 5000 insects of type A and 3000 insects of type B.

a On the same set of axes sketch the graphs, $A(t)$ and $B(t)$ for $0 \leq t \leq 4$.

b What is the maximum number of insects of type A that will occur?

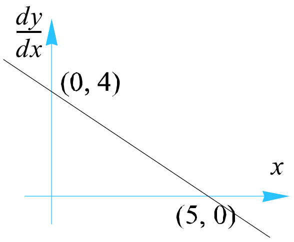
c When will there first be equal numbers of insects of both types?

d For how long will the number of type B insects exceed the number of type A insects during the four days?

Exercise E.12.4

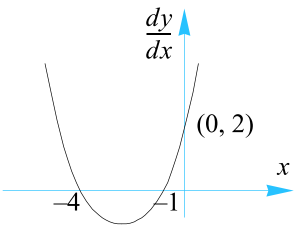
11. Sketch the graph of $y = f(x)$ for each of the following:

a



Where the curve passes through the point (5, 10).

b



Where the curve passes through the point (0, 0).

12. Find $f(x)$ given that $f''(x) = 12x + 4$ and that the gradient at the point (1, 6) is 12.

13. Find $f(x)$ given that $f'(x) = ax^2 + b$, where the gradient at the point (1, 2) is 4, and that the curve passes through the point (3, 4).

14. The rate at which a balloon is expanding is given by

$$\frac{dV}{dt} = kt^{4.5}, t \geq 0,$$

where t is the time in minutes since the balloon started to be inflated and V cm³ is its volume. Initially the balloon (which may be assumed to be spherical) has a radius of 5 cm. If the balloon has a volume of 800 cm³ after 2 minutes, find its volume after 5 minutes.

15. The area, A cm², of a healing wound caused by a fall on a particular surface decreases at a rate given by the equation:

$$A'(t) = -\frac{35}{\sqrt{t}}$$

where t is the time in days. Find the initial area of such a wound if after one day the area measures 40 cm².

Exercise E.12.5

6. Evaluate the following definite integrals (giving exact values).

g $\displaystyle\int_0^1 \frac{2}{(x+1)^3}dx$ h $\displaystyle\int_2^4 \left(\sqrt{x}-\frac{2}{\sqrt{x}}\right)^2 dx$ i $\displaystyle\int_3^4 \frac{2x+1}{2x^2-3x-2}dx$

8. Evaluate the following definite integrals (giving exact values).

e $\displaystyle\int_0^{\frac{\pi}{4}} (x-\sec^2 x)dx$ f $\displaystyle\int_0^{\frac{\pi}{2}} 2\cos\left(4x+\frac{\pi}{2}\right)dx$ g $\displaystyle\int_{-\pi}^{\pi} \left(\sin\left(\frac{x}{2}\right)+2\cos(x)\right)dx$

h $\displaystyle\int_0^{\frac{\pi}{12}} \sec^2\left(\frac{\pi}{4}-2x\right)dx$ i $\displaystyle\int_0^{\pi} \cos(2x+\pi)dx$

13. a Find $\frac{d}{dx}(xe^{0.1x})$. Hence, find $\int xe^{0.1x}dx$.

b Following an advertising initiative by the Traffic Authorities, preliminary results predict that the number of alcohol-related traffic accidents has been decreasing at a rate of $-12-te^{0.1t}$ accidents per month, where t is the time in months since the advertising campaign started.

i How many accidents were there over the first six months of the campaign?

ii In the year prior to the advertising campaign there were 878 alcohol-related traffic accidents. Find an expression for the total number of accidents since the start of the previous year, t months after the campaign started.

14. The rate of cable television subscribers in a city t years from 1995 has been modelled by the equation $\frac{2000}{\sqrt{(1+0.4t)^3}}$.

a How many subscribers were there between 1998 and 2002?

b If there were initially 40 000 subscribers, find the number of subscribers by 2010.

15.

a Find $\frac{d}{dt}\left(\frac{800}{1+24e^{-0.02t}}\right)$.

b The rate at which the number of fruit flies appear when placed in an environment with limited food supply in an experiment was found to be approximated by the exponential model:

$$\frac{384e^{-0.02t}}{(1+24e^{-0.02t})^2},\ t \geq 0$$
,

where t is the number of days since the experiment started. What was the increase in the number of flies after 200 days?

Exercise E.12.6

20. Find the area of the region bounded by the curves with equations $y = \sqrt{x}$, $y = 6 - x$ and the x-axis.

21. a Sketch the graph of the function $f(x) = |e^x - 1|$.

b Find the area of the region enclosed by the curve $y = f(x)$,

i the x-axis and the lines $x = -1$ and $x = 1$.

ii the y-axis and the line $y = e - 1$.

iii and the line $y = 1$. Discuss your findings for this case.

22. a On the same set of axes, sketch the graphs of $f(x) = \sin\left(\frac{1}{2}x\right)$ and $g(x) = \sin 2x$ over the interval $0 \le x \le \pi$.

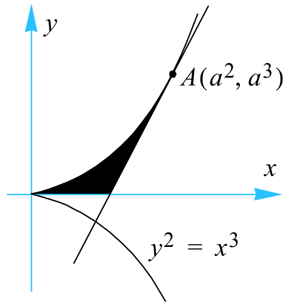
b Find the area of the region between by the curves $y = f(x)$ and $y = g(x)$ over the interval $0 \le x \le \pi$, giving your answer correct to two decimal places.

23. Consider the curve with equation $y^2 = x^3$ as shown in the diagram.

A tangent meets the curve at the point $A(a^2, a^3)$.

a Find the equation of the tangent at A.

b Find the area of the shaded region enclosed by the curve, the line $y = 0$ and the tangent.



24. a On a set of axes, sketch the graph of the curve $y = e^{x-1}$ and find the area of the region enclosed by the curve, the x-axis and the lines $x = 0$ and $x = 1$.

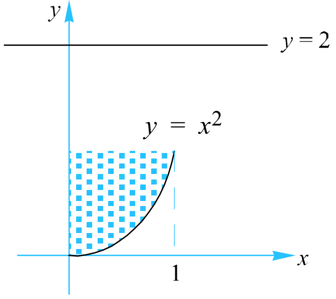
b Hence evaluate $\int_{e^{-1}}^{1} (\ln x + 1)dx$.

c Find the area of the region enclosed by the curves $y = e^{x-1}$ and $y = \ln x + 1$ over the $e^{-1} \le x \le 1$.

Exercise E.12.7

Extension problems

23. Find the volume of the solid of revolution generated when the shaded region shown below is revolved about the line $y=2$.



24. Find the volume of the solid of revolution generated by revolving the region enclosed by the curve $y = 4 - x^2$ and $y = 0$ about:

a the line $y = -3$

b the line $x = 3$

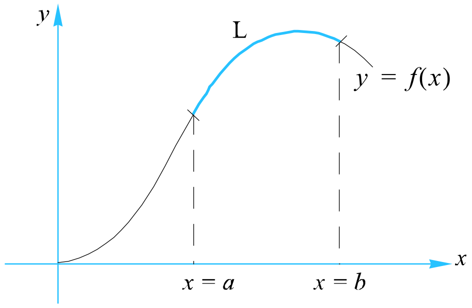
c the line $y = 7$.

In each case, draw the shape of the solid of revolution.

25. Show why the arc length, L units, of a curve from:

$x = a$ to $x = b$ is given by:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}dx.$$



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